Existence of equilibria in economies with externalities and non-convexities in an infinite dimensional commodity space[1]

Matías N. Fuentes[2]

Universidad Nacional de San Martín, CIETyMA, Caseros 2241 (1650), Buenos Aires, Argentina.

Abstract

We prove an equilibrium existence theorem for economies with externalities, general types of non-convexities in the production sector, and infinitely many commodities. The consumption sets, the preferences of the consumers and the production possibilities are represented by set-valued mappings to take into account the external effects. The firms set their prices according to general pricing rules which are supposed to have bounded losses and may depend upon the actions of the other economic agents. The commodity space is $L_\infty(M, \mathcal{M}, \mu)$, the space of essentially bounded, real-valued, measurable functions on $(M, \mathcal{M}, \mu)$.

As for our existence result, we consider the framework of Bewley (1972). However, there are four major problems in using this technique. To overcome two of these difficulties, we impose strong lower hemi-continuity assumptions upon the economies. The remaining problems are removed when finite economies are large enough.

Our model encompasses previous works on the existence of general equilibria when there are externalities and non-convexities but the commodity space is finite dimensional and those on general equilibria in non-convex economies with infinitely many commodities when no external effect is taken into account.

Keywords: General equilibrium; Externalities; Non-convexities; Infinitely many commodities; Set-valued mappings; Lower hemi-continuity.

1. Introduction

This paper deals with three levels of extensions from the standard competitive equilibrium model: the presence of externalities, the possibility of increasing returns to scale or more general types of non-convexities in production, and an infinite dimensional commodity space of goods. The purpose is to give sufficient conditions for the existence of an equilibrium.

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[2] E-mail address: matias.fuentes@unsam.edu.ar. Phone: 00 11 4580 – 7250/51 int. 111. Fax: 00 11 4580-7250 int. 107

Although usually the economic literature considers that the non-convexities are coming from indivisibilities, increasing returns to scale, or fixed costs, it is also recognized that external effects are sources of non-convexities in production. We refer to the examples in Mas-Colell *et al* (1995, pp. 374-377), which illustrate that externalities may themselves generate non-convexities.

In economies with a finite dimensional commodity space like $\mathbb{R}^l$, both Bonnisseau (1997) as well as Bonnisseau and Médecin (2001) have achieved important results. The first one considers a general equilibrium model in economies with non-convex firms, where production sets may depend on the production plans of other firms and on consumption plans, but also the consumption sets and the preferences of the consumers may change with respect to the environment. His main theorem proves the existence of general equilibria under assumptions which allows him to encompass together the works on the existence of competitive equilibria with externalities when the firms have convex production sets and on the existence of equilibria with general pricing rule without externalities as in Bonnisseau and Cornet (1988). Bonnisseau and Médecin (2001) define the marginal pricing rule as the convex hull of the limits of the prices given by the normal cone in a neighborhood of the production plan. They use the same procedure as the one used to define the Clarke´s normal cone (see Clarke (1983)) and the marginal pricing rule without externalities. Their framework is sufficiently large to generalize previous works on economies with externalities and convex production sets and those on marginal pricing equilibria in economies without externalities as in the model of Bonnisseau and Cornet (1990b). In both papers, the production possibilities, the consumption sets, and the preferences of the consumers, are represented by set-valued mappings which associate the set of possible decisions plans to a given environment. This means that both consumers and producers take into account the fact that their decisions depend on external effects.

With regard to the infinite dimensional space, it is related to the possibilities of having infinitely many commodities. This arises naturally when one considers economic activity over an infinite time horizon, or with uncertainty about the infinite number of states of the world, or in a setting where an infinite variety of commodity characteristics are possible. Bewley (1972) proved the first infinite dimensional equilibrium existence theorem with production. The commodity space he treated was $L_\infty\left(M,\mathcal{M},\mu\right)$, the space of essentially bounded, real-valued, measurable functions on $\left(M,\mathcal{M},\mu\right)$. The strategy of Bewley´s proof was to consider the restriction of the original economy $\mathcal{E}$ to finite-dimensional sub-economies $\mathcal{E}^F$, where F is a finite dimensional subspace of $L_\infty\left(M,\mathcal{M},\mu\right)$ which contains the initial endowments. Standard results imply that each of the sub-economies $\mathcal{E}^F$ has an equilibrium. Then, Bewley proved that an equilibrium for the original economy can be obtained as a limit of equilibria for the finite dimensional economies. The extension of his result to infinite dimensional spaces other than L_∞, presents the problem of the emptiness of the interior of the positive cone. In this case, the existence of functionals and supporting prices is not guaranteed. A solution to this problem, when the commodity space is a topological vector lattice, was presented in a seminal paper by Mas-Colell (1986). His solution was based on the assumption that preferences satisfy a cone condition that he termed uniform properness, an adaptation of a condition that was used in the work of Chichilnisky and Kalman (1980). The properness condition was studied also in Yannelis and Zame (1986), Zame (1987) and Richard (1989) among others.

Bonnisseau and Meddeb (1999) have extended the work of Bewley (1972) to allow the existence of increasing returns to scale or more general types of non-convexities in production. They obtained an existence result under assumptions which also extended to those considered in the finite dimensional case. Later Bonnisseau (2002) proposed a new definition of the marginal pricing rule which allowed him to encompass the cases of smooth and convex production sets for which his definition of the marginal pricing rule coincides with the one given by the Clarke´s normal cone or the normal cone of convex analysis. Bonnisseau also showed the link with the definition used in a finite dimensional setting where the marginal pricing rule is also defined by means of the Clarke´s normal cone. Both Bonnisseau and Meddeb (1999) as well as Bonnisseau (2002) used the Bewley approach. However, and in contrast to Bewley´s proof, an equilibria may not be found for every finite dimensional economy. The problem is that even the original economy is supposed to satisfy both survival and local non-satiation assumptions; this may not be true for the auxiliary economies. Thus, in Bonnisseau and Meddeb (1999) the assumptions made on the original economy lead to an equilibrium at the limit of some subnet whose elements may not be an equilibria of the restricted economies. In turn, Bonnisseau (2002) showed that both assumptions hold true if the commodity space is large enough. Then, he applied the limit argument to a net of equilibria vectors of the induced economies.

As for our existence result, we also consider the framework of Bewley (1972). However, there are four major problems in using this technique in order to prove the existence of a general equilibrium when we consider together externalities, non-convexities of production sets, and infinite dimensional commodity spaces. The first problem is that the lower hemi-continuity of a correspondence may not hold when we restrict the commodity space to a finite dimensional subspace. In our model, this means that it is not possible to utilize the existing existence theorems of Bonnisseau (1997). We remark that the lower hemi-continuity of the consumption and production set-valued mappings would be the natural generalization, to the infinite dimensional space, of the assumptions made in Bonnisseau (1997). The second problem is that, even if there is an equilibria in each $\mathcal{E}^F$, the lower hemi-continuity of the consumption set-valued mappings is not enough to guarantee that the limit point is an equilibria for the original economy. The third problem is that the so-called survival assumption may not hold when we restrict the commodity space to a finite dimensional subspace. Again, this means that we cannot apply the finite dimensional results of Bonnisseau (1997) to $\mathcal{E}^F$. The final difficulty comes from the fact that, even if the original economy is supposed to satisfy the local non-satiation assumption, this may not be true for the auxiliary economies $\mathcal{E}^F$. The consequences are the same as those in the first and the third problem.

To avoid the first difficulty we need to posit additional assumptions on the restricted set-valued mappings whereas to solve the second problem we must impose a strong lower hemi-continuity assumption on the consumption correspondences. For the third problem, we work on the finite dimensional economy with a weak version of the survival assumption of Bonnisseau (1997) by considering only the elements of production equilibria, which are not too far from the attainable allocations. Finally, we show that the local non-satiation assumption holds true, on the attainable allocations, if the commodity space is large enough.

Other difficulties that appear are related with the behavior of producers, that is, the pricing rule correspondences, and the continuity of the distribution wealth functions of the different agents. These are the same difficulties that appear in Bonnisseau and Meddeb (1999) and we refer to their work.

The paper is organized as follows: In section 2, we introduce the model and the notations to deal with externalities, non-convexities, and infinite dimensional spaces. In Section 3 we give the basic assumptions and we discuss them. In section 4 we first define the finite dimensional auxiliary economies and we posit an assumption on the pricing rules. Then, we discuss Bewley´s limiting technique in our model and we introduce new lower hemi-continuity assumptions upon the consumption and production correspondences. In section 5 we state the existence result. Section 6 is devoted to the proof of the existence result. The proofs of technical lemmas are given in Appendix A whereas Appendix B deals with the special case of producers having convex production sets.

2. The model

We consider an economy with an infinite dimensional commodity space represented by the space of essentially bounded, real-valued, measurable functions on a σ–finite positive measure space (M, $\mathcal{M}$, μ). For a survey on the properties of the space $L = L_\infty(M, \mathcal{M}, \mu)$[3] and its use in general equilibrium theory, we refer to Mas-Colell and Zame (1991).

There is a finite number $i = 1, \ldots, m$, of consumers, and a finite number $j = 1,\ldots,n$, of producers. As much as possible, we will use the notation, definitions and concepts of Bonnisseau (1997) and Bonnisseau and Meddeb (1999): Let $z = \left((x_i)_{i=1}^m, (y_j)_{j=1}^n \right) \in L^{m+n}$; for all i, z_{-i} is the element of L^{m+n-1} defined by

$$z_{-i} = \left((x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m), (y_j)_{j=1}^n \right)$$

and, for all j, z_{-j} is the element of L^{m+n-1} defined by

$$z_{-j} = \left((x_i)_{i=1}^m, (y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \right)$$

Following Mas-Colell *et al* (1995) an *externality* is present whenever the well-being of an economic agent is directly affected by the actions of the others agents. "Directly" means that we exclude any effect that is mediated by prices. Formally, the *consumption set* of the *i*-th consumer is represented by a set-valued mapping X_i from L^{m+n-1} to L_+. For the environment $z_{-i} \in L^{m+n-1}$, $X_i(z_{-i}) \subset L_+$ is the set of possible consumption plans of the *i*-th consumer. Analogously, the *production set* of the *j*–th producer is defined by a set-valued mapping Y_j from L^{m+n-1} to L. $Y_j(z_{-j})$ is the set of all feasible production plans for the *j*–th firm when the actions of the other economic agents are given by z_{-j}.

A *price system* is a continuous linear mapping on L_+. If L_+ is endowed with the norm topology, the set of prices is $L_+^* = ba_+(M, \mathcal{M}, \mu)$, the space of bounded additive set functions on (M, $\mathcal{M}$) absolutely continuous with respect to μ. Thus, the value of a *commodity bundle* $x \in L_\infty$ is $\int_M x d\pi$ (see Dunford and Scwhartz (1958)). Note that the elements of $ba(M, \mathcal{M}, \mu)$ belonging to $L_1(M, \mathcal{M}, \mu)$ have, additionally, an economic interpretation since the value of a commodity bundle $x \in L_\infty$ at a price system $p \in L_1^+$ is $\int_M p(m) x(m) d\mu(m)$ which is the natural generalization of the concept of inner product in the finite dimensional case.

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[3] Without any loss of generality we directly call to $L_\infty(M, \mathcal{M}, \mu)$ as the set of equivalence classes of all μ-essentially bounded $\mathcal{M}$–measurable functions on M. Let x be an element of $L_\infty(M, \mathcal{M}, \mu)$, then $x \geq 0$ if $x(m) \geq 0$ μ-a.e. (almost everywhere); $x > 0$ if $x \geq 0$ and $x \neq 0$, and $x >> 0$ if $x(m) > 0$ μ-a.e. Hence, if $x, x' \in L_\infty(M, \mathcal{M}, \mu)$ then $x \geq x'$ (respectively $x > x'$, $x >> x'$) if $x - x' \geq 0$ (respectively $x - x' > 0$, $x - x' >> 0$). $L_+ = \{x \in L : x \geq 0\}$ is the positive cone of L

$S = \{\pi \in ba_+(M, \mathcal{M}, \mu) : \pi(\chi_M) = 1\}$ is the price simplex, where χ_M is the function equal to 1 for every m in M.

The weak-star topology $\sigma(L, L_1) = \sigma^\infty$ is the weakest topology for which the topological dual of L is L_1. We denote by $\prod_{L^s} \sigma^\infty$ the product topology on L^s. $\sigma(L, ba)$ and $\sigma(ba, L) = \sigma^{ba}$ are the weak and the weak-star topologies, respectively, on L and ba. Let $A : L^s \mapsto L$ be a set-valued mapping. We say that A is $\left(\prod_{L^s} \sigma^\infty, \sigma^\infty\right)$-closed if it has a closed graph for the product of weak-star topologies. We denote by T the norm topology on L. The set-valued mapping A is said to be $\left(\prod_{L^s} \sigma^\infty, \mathrm{T}\right)$–lower hemi-continuous (for short l.h.c.) if for every net (z^α) in L^s which converges to z in $\prod_{L^s} \sigma^\infty$ and $a \in A(z)$, there is a net (a^α) such that $a^\alpha \in A(z^\alpha)$ for all α and a^α converges to a in T. For more information about topological vector spaces we refer to Schaefer (1971), and Aliprantis and Border (1994).

The tastes of the consumers are described by a complete, reflexive, transitive, binary *preference relation* $\sum_{i,z_{-i}}$ on the set $X_i(z_{-i})$. We do not assume that a consumer has a preference between two consumption plans if the environment is different.

Let $\omega_i \in L_+$ be the *initial endowment* of the *i*-th agent and $\omega = \sum_{i=1}^m \omega_i$ the *total initial endowment* of the economy. The *revenue* of the *i*-th agent is defined by a wealth function from $\sim^{1+n}$ to $\sim$. $r_i\left(\pi(\omega_i), \left(\pi(y_j)\right)_{j=1}^n\right)$ is the wealth of the *i*-th agent if the price vector is $\pi \in S$ and the production plans are $(y_j)_{j=1}^n \in \prod_{j=1}^n Y_j(z_{-j})$. Note that the revenues of the agents depend on the value of the initial endowments and the profit or losses of the producers.

The set of *weakly efficient allocations* is

$$Z = \left\{ z \in L^{m+n} : \forall i \; x_i \in X_i(z_{-i}), \forall j \; y_j \in \partial_\infty Y_j(z_{-j}) \right\}$$

where $\partial_\infty Y_j(z_{-j})$ is the boundary of $Y_j(z_{-j})$ for the norm topology.

We now describe the behavior of the producers. The *j*-th firm follows the *pricing rule* φ_j defined by a set-valued mapping from *Z* to *S*. If $z \in Z$, $\pi \in \varphi_j(z)$ is the price that the *j*-th producer sets on the market. Hence, the *j*-th firm is in equilibrium at the pair $(\pi, z) \in S \times Z$ if $\pi \in \varphi_j(z)$. The use of pricing rules to define the behavior of the producers is natural in non-convex economies as it is explained in Cornet (1988).

When the *j*-th producer maximizes his profit given the price system π, his pricing rule is given by

$$\varphi_j(z) = PM_j(z) = \left\{\pi \in S : \pi\left(y_j\right) \geq \pi(y), \forall\, y \in Y_j\left(z_{-j}\right)\right\}$$

The set of *weakly efficient attainable allocations* corresponding to a given total initial endowment is

$$A(\omega) = \{z \in Z : \textstyle\sum_{i=1}^m x_i \leq \sum_{j=1}^n y_j + \omega\}$$

Finally, the set of *production equilibria* is

$$PE = \left\{(\pi, z) \in S \times Z : \pi \in \textstyle\bigcap_{j=1}^n \varphi_j(z)\right\}$$

We now define our notion of equilibrium

Definition 2.1 $(\bar{z}, \bar{\pi}) = \left(\left(\left(\bar{x}_i\right)_{i=1}^m, \left(\bar{y}_j\right)_{j=1}^n\right), \bar{\pi}\right) \in Z \times S$ is an equilibrium of the economy $\mathcal{E} = \left(\left(X_i, \sum_{i,z_{-i}}, r_i\right)_{i=1}^m, \left(Y_j, \varphi_j\right)_{j=1}^n, \left(\omega_i\right)_{i=1}^m\right)$ if:

a. For all i, $\bar{x}_i$ is $\sum_{i,\bar{z}_{-i}}$-maximal in $\left\{x_i \in X_i\left(\bar{z}_{-i}\right) : \bar{\pi}\left(x_i\right) \leq r_i\left(\bar{\pi}\left(\omega_i\right), \left(\bar{\pi}\left(\bar{y}_j\right)\right)_{j=1}^n\right)\right\}$

b. $\bar{\pi} \in \bigcap_{j=1}^n \varphi_j(\bar{z})$

c. $\sum_{i=1}^m \bar{x}_i = \sum_{j=1}^n \bar{y}_j + \omega$

Condition a. says that every consumer maximizes his preference under his budget constraint. Condition b. says that every producer sets on the market the same equilibrium price vector $\bar{\pi}$. Condition c. says that all markets clear.

3. Basic Assumptions

Assumption (C): About consumers

For all i

(i) X_i is a $\left(\prod_{L^{m+n-1}} \sigma^\infty, \sigma^\infty\right)$-closed set valued mapping with convex values and containing 0.

(ii) Preferences are non-satiated, that is, for all $z_{-i} \in L^{m+n-1}$, and for all x_i in $X_i\left(z_{-i}\right)$, there exists x in $X_i\left(z_{-i}\right)$ such that $x_i \prec_{i,z_{-i}} x$. They are also convex, that is, for all $\left(x_i, x_i'\right) \in X_i\left(z_{-i}\right)^2$, and for all $t \in (0,1)$, if $x_i \prec_{i,z_{-i}} x_i'$ then $x_i \prec_{i,z_{-i}} tx_i + (1-t)x_i'$.

(iii) The set $\Gamma_i = \left\{\left(z_{-i}, x_i, x_i'\right) \in L^{m+n+1} : \left(x_i, x_i'\right) \in X_i\left(z_{-i}\right)^2, x_i \,\partial_{\,i,z_{-i}}\, x_i'\right\}$ is closed in L^{m+n+1} for the product of weak-star topologies.

(iv) The function r_i is continuous. It is also strictly increasing in the second variable. Furthermore, $\sum_{i=1}^{m} r_i\left(\pi(\omega_i),\ \left(\pi(y_j)\right)_{j=1}^{n}\right) = \pi(\omega) + \sum_{j=1}^{n} \pi(y_j)$

Assumption (P): About producers

For all j

(i) Y_j is a $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, \sigma^{\infty}\right)$-closed set valued-mapping.

(ii) Y_j is a $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, \mathcal{T}\right)$-l.h.c. set valued-mapping.

(iii) There exist $\underline{t}_j$ and $\overline{t}_j$ in $\mathbb{R}$ such that for all $z_{-j} \in L^{m+n-1}$, $\underline{t}_j \chi_M \in Y_j(z_{-j})$ and $\overline{t}_j \chi_M \notin Y_j(z_{-j})$

(iv) $Y_j(z_{-j})$ satisfies the free disposal condition, that is, $Y_j(z_{-j}) - L_+ = Y_j(z_{-j})$ for all $z_{-j} \in L^{m+n-1}$.

Assumption (B)

For all $\omega' \geq \omega$, the set

$A(\omega') = \{z \in Z : \sum_{i=1}^{m} x_i \leq \sum_{j=1}^{n} y_j + \omega'\}$ is norm bounded.

We remark that in an economy without externalities, that is if X_i and Y_j are set-valued mappings with constant values for all i, j, and if $\sum_i$ is a constant preference relation for each i, the above assumptions are the same as those in Bonnisseau and Meddeb (1999)[4]. We also remark that if the commodity space is $\sim^{l}$, then Assumptions (C), (P) and (B) coincide with the ones in Bonnisseau (1997)[5].

C(i) is the natural extension of the assumptions concerning consumers to consider externalities. It implies that, for all $z_{-i} \in L^{m+n-1}$, the consumption set is non-empty and bounded below by 0. In C(ii) we remark that the convexity of the preferences implies that they are locally non satiated (LNS) if they are non-satiated, that is, for all $z_{-i} \in L^{m+n-1}$, for all x_i in $X_i(z_{-i})$, and for all $\varepsilon > 0$, there exists x in $X_i(z_{-i}) \cap B(x_i, \varepsilon)$[6] such that $x_i \prec_{i, z_{-i}} x$. Assumption C(iii) deals with the continuity of the preference relations with respect to the environment and to the consumption bundles. In C(iv) the assumptions made on the wealth functions are similar to the ones used in Bonisseau and Meddeb (1999).

[4] Except for P(ii) which is not required.

[5] Except for the Assumption C(ii) in the paper of Bonnisseau that will be discussed later. We also remark that the assumptions $X_i(z_{-i}) \subset \sim^{l}_{+}$ and $0 \in X_i(z_{-i})$ are not in Bonnisseau (1997).

[6] $B(x, \varepsilon)$ is the open ball of center x and radius ε.

There is no convexity assumption on the firms, neither on the individual production sets nor on the global one. P(i) is the extension of the assumption that every production set is closed by considering externalities. P(ii) generalizes the lower hemi-continuity assumption of Bonnisseau (1997) to infinite dimensional spaces. P(iii) is not assumed in Bonnisseau and Meddeb (1999) but in Bonnisseau (1997). We will need it when considering the finite dimensional economy. P(iv) is the well-known free disposal condition which merely says that if y is producible, then it is always possible to produce y' so that y' produces at most the same amount of outputs using at least the same amount of inputs. On the other hand, we remark that under assumption (P), for every *j*, the set of weakly efficient production plans of the *j*-th producer is exactly $\partial_\infty Y_j\left(z_{-j}\right)$.

Assumption (B) supposes a given boundary for all $\omega' \geq \omega$ and not just for ω. This assumption is clearly stronger than simply assuming that the set of attainable allocations is bounded since it means that the attainable allocations are bounded even if one increases the initial endowments. We need to consider this assumption since in Bonnisseau and Cornet (1988) there is an example of an economy without equilibrium since Assumption B is not satisfied (see remark 5.1 in their paper).

Assumption (BL) (bounded loss)

For all *j*, there exists a real number α_j such that: $z \in Z$ and $\pi_j \in \varphi_j\left(z\right)$ imply that $\pi_j\left(y_j\right) \geq \alpha_j$

Assumption (WSA) (Weak survival assumption)

For all $\left(\pi, z, \lambda\right) \in PE \times \sim_{+}$, if $z \in A\left(\omega + \lambda\chi_M\right)$ then $\pi\left(\sum_{j=1}^{n} y_j + \omega + \lambda\chi_M\right) > 0$

Assumption (R)

For all $\left(\pi, z\right) \in PE$, if $z \in A\left(\omega\right)$ then $r_i\left(\pi\left(\omega_i\right), \left(\pi\left(y_j\right)\right)_{j=1}^{n}\right) > 0$.

Assumption (BL) means that the loss of the *j*-th firm is bounded below when its pricing rule is φ_j. As for Assumption WSA, note that if $z \in A\left(\omega + \lambda\chi_M\right)$ then $\sum_{j=1}^{n} y_j + \omega + \lambda\chi_M \geq 0$. Consequently, $\pi\left(\sum_{j=1}^{n} y_j + \omega + \lambda\chi_M\right) \geq 0$ for every $\pi \in L_+^*$. Therefore, we just require that the inequality is strict, that is, when the same price is offered by the producers, according to their pricing rules, the global wealth of the economy is strictly greater than the minimum possible. We recall that it is satisfied when the production sets are convex and the initial endowments are in the interior of L_+. We also note that (WSA) is weaker than the survival assumption used in Bonnisseau and Cornet (1988) (see Kamiya (1988)). Finally, Assumption (R) can be interpreted as the fact that each consumer has a positive income if the total wealth of the economy is positive.

4. Sub-economies, Bewley´s limiting approach and additional assumptions.

In this section we analyze the finite dimensional economies or sub-economies in order to apply the method of finite approximations in section 6. In sub-section 4.1 below we first define these sub-economies and then we posit an assumption on the pricing rule correspondences. In sub-section 4.2, we point out the difficulties that arise when we apply Bewley´s technique to our model. Then we introduce new lower hemi-continuity assumptions to overcome two of these difficulties.

4.1 S*ub-economies and a continuity assumption on the pricing rule correspondences*

Let F be a finite dimensional subspace of L such that $\omega_i \in F$ for all i and $\chi_M \in F$. Let $\mathcal{F}$ denote the family of all such F. The family $\mathcal{F}$ is directed by the set inclusion. For every $F \in \mathcal{F}$, $F_+ = F \cap L_+$ is the positive cone of F and int$F_+ = F \cap$intL_+ is the interior of F_+ which is non-empty since χ_M belongs to the norm interior of L_+. Each $F \in \mathcal{F}$ is equivalent to a finite dimensional Euclidean space (Dunford & Schwartz (1958), p. 245). Since F is a proper convex cone, these two facts allow us to choose an Euclidean structure on F such that $\| \chi_M \| = 1$ and the orthogonal space to χ_M, $\chi_M^{\perp F}$, satisfies $\left\{\chi_M^{\perp F}\right\} \cap F_+ = \{0\}$. We remark that χ_M plays the role of e in the work of Bonnisseau (1997). Finally, F^* is the dual space of $F \in \mathcal{F}$.

For every $F \in \mathcal{F}$, let us consider the sub-economy

$$\mathcal{E}^F = \left(\left(X_i^F, \sum\nolimits^F_{i, z_{-i}^F}, r_i^F \right)_{i=1}^m, \left(Y_j^F, \varphi_j^F \right)_{j=1}^n, \left(\omega_i \right)_{i=1}^m \right)$$

Where, for all i and j, X_i^F and Y_j^F are the restricted set valued-mappings:

$$X_i^F : F^{m+n-1} \mapsto 2^{F_+}$$

$$Y_j^F : F^{m+n-1} \mapsto 2^F$$

defined by

$$X_i^F \left(z_{-i}^F \right) = X_i \left(z_{-i}^F \right) \cap F_+$$

$$Y_j^F \left(z_{-j}^F \right) = Y_j \left(z_{-j}^F \right) \cap F$$

Let $S^F = \left\{ p^F \in F_+^* : p^F \left(\chi_M \right) = 1 \right\}$ be the price simplex of $\mathcal{E}^F$. r_i^F is the revenue of the i-th consumer induced by r_i on the restricted economy. The relation ∂^F_{i, z_{-i}^F} is the preordering induced on $X_i^F \left(z_{-i}^F \right)$ by ∂_{i, z_{-i}^F}.

We remark that for all $F \in \mathcal{F}$ and all i, assumption C(i) together with the fact that F is a subspace of L imply that $X_i^F \left(z_{-i}^F \right)$ is non-empty. In the same way, for all $F \in \mathcal{F}$ and all j, P(iii) together with the fact that F is a subspace of L imply that $Y_j^F \left(z_{-j}^F \right)$ is non-empty .

Note that for all $F \in \mathcal{F}$ and all $z_{-j}^F \in F^{m+n-1} \subset L^{m+n-1}$, $\partial\left(Y_j\left(z_{-j}^F\right) \cap F\right) \subset \partial_\infty Y_j\left(z_{-j}^F\right)$. Consequently, if we define the set of weakly efficient allocations of the sub-economy by

$$Z^F = \left\{z^F \in F^{m+n} : \forall i\ x_i^F \in X_i^F\left(z_{-i}^F\right), \forall j\ y_j^F \in \partial Y_j^F\left(z_{-j}^F\right)\right\}$$

then $Z^F \subset Z$. Hence, for all j and all $z^F \in Z^F$ we can define

$$\varphi_j^F\left(z^F\right) = \left\{p^F \in S^F : \text{there exists } \pi \in \varphi_j\left(z^F\right) \text{ and } p^F = \pi_{|F}\right\}.$$

Note that if $\varphi_j\left(z^F\right)$ is non-empty, then so is $\varphi_j^F\left(z^F\right)$.

The set of production equilibria in $\mathcal{E}^F$ is

$$PE^F = \left\{(p^F, z^F) \in S^F \times Z^F : p^F \in \bigcap_{j=1}^n \varphi_j^F\left(z^F\right)\right\}$$

Finally, the set of weakly efficient attainable allocations in $\mathcal{E}^F$ is

$$A^F(\omega') = \left\{z^F \in Z^F : \sum_{i=1}^m x_i^F \leq \sum_{j=1}^n y_j^F + \omega'\right\} \subset A(\omega')$$

We can now posit the following assumption on the behavior of the producers.

Assumption (PR)

For all j

(i) The correspondence $\varphi_j : Z \mapsto 2^S$ is non-empty and convex valued.
(ii) For every F in $\mathcal{F}$, the correspondence φ_j^F has a closed graph.
(iii) Let $\left(z^{F(t)}, \pi^{F(t)}\right)_{t \in T}$ be a subnet of a net $\left(z^F, \pi^F\right)_{F \in \mathcal{F}} \in Z \times S$, such that

$$\begin{cases} \left(z^{F(t)}, \pi^{F(t)}\right) \to \left(\bar{z}, \bar{\pi}\right) \text{ for the product of weak-star topologies} \\ \pi^{F(t)} \in \varphi_j\left(z^{F(t)}\right) \text{ for all } t \in T \\ \pi^{F(t)}\left(y_j^{F(t)}\right)_{t \in T} \text{ converges} \end{cases}$$

We have:

(a) $\lim \pi^{F(t)}\left(y_j^{F(t)}\right) \geq \bar{\pi}\left(\bar{y}_j\right)$,

Furthermore, if $\lim \pi^{F(t)}\left(y_j^{F(t)}\right) = \bar{\pi}\left(\bar{y}_j\right)$, then

(b) $\bar{y}_j \in \partial_\infty Y_j\left(\bar{z}_{-j}\right)$ and $\bar{\pi} \in \varphi_j\left(\bar{z}\right)$

We remark that if there is no externality, this assumption coincides with the Assumption PR of Bonnisseau and Meddeb (1999). We also note that if the commodity space is $\sim^{l}$, then our assumption is the same as the Assumption PR of Bonnisseau (1997). Finally, if Y_j is convex-valued and satisfies Assumption (P), then $\varphi_j = PM_j$ satisfies Assumption (PR) (see appendix B).

4.2 *Bewley´s limiting technique and additional lower hemi-continuity assumptions*

As for the technique of proof, we follow (as we have already said before) the Bewley (1972) approach. The proof is similar since we consider a net of equilibria of sub-economies with increasing finite dimensional commodity spaces. Unfortunately there are four major problems in using Bewley´s technique for such sub-economies. First, if we simply assume that the correspondences X_i and Y_j are l.h.c. for all i and j (which would be the natural generalization of the lower hemi-continuity assumptions of Bonnisseau (1997)), the restriction to a finite dimensional subspace may not always be l.h.c. (see Sun (2006), example 1). As a result, the necessary conditions for the existence of an equilibrium given in Bonnisseau (1997) are not necessarily fulfilled by the sub-economies $\mathcal{E}^F$.

The second problem is that, even if there is an equilibrium in each $\mathcal{E}^F$, it is impossible to prove that a limit point $\left(\left(\left(\bar{x}_i\right)_{i=1}^{m}, \left(\bar{y}_j\right)_{j=1}^{n}\right), \bar{\pi}\right)$ is an equilibrium in the original infinite dimensional economy. The crucial point is that the lower hemi-continuity of X_i is not enough to prove that, for all i, if $x_i \sum_{i,\,\bar{z}_{-i}} \bar{x}_i$ then $\bar{\pi}\left(x_i\right) \geq r_i\left(\bar{\pi}\left(\omega_i\right), \left(\bar{\pi}\left(\bar{y}_j\right)\right)_{j=1}^{n}\right)$ (see steps 4 and 5 together with the remark 6.1 below) .

Kajii (1988) and Sun (2006) have investigated the lower hemi-continuity problems and proposed alternative solutions to avoid these difficulties. Although they dealt with correspondences for preference relations, the mathematical aspects of the problems are identical to ours. With regard to the first problem, the solution of Kajii implies assuming that the restriction of a set-valued mapping to a finite dimensional subspace is l.h.c. In turn, relative to the second problem, the solution of Sun implies making use of a strong lower hemi-continuity assumption that he called $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, f\right)$-l.h.c. with the notations of this paper. We show that we can avoid both problems by adapting their assumptions

Assumption (C)

For all i

(v) There is a finite dimensional subspace $\bar{F} \in \mathcal{F}$, such that for any finite dimensional subspace $F \in \mathcal{F}$ such that $\bar{F} \subset F$, the set-valued mapping X_i^F is l.h.c. on F^{m+n-1} .

(vi) The set-valued mapping X_i is $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, f\right)$-l.h.c. on L^{m+n-1},that is, if $z_{-i}^{\alpha} \longrightarrow z_{-i}$ in L^{m+n-1} for the product of weak-star topologies and $x \in X_i\left(z_{-i}\right)$, there exists a finite dimensional subspace $\dot{F}$ such that there is a net $\left(x^{\alpha}\right) \subset x + \dot{F}$ with $x^{\alpha} \in X_i\left(z_{-i}^{\alpha}\right)$ for all α and $x^{\alpha} \longrightarrow x$.

Assumption (P)

For all *j*

(v) There is a finite dimensional subspace $\overline{\overline{F}} \in \mathcal{F}$, such that for any finite dimensional subspace $F \in \mathcal{F}$ such that $\overline{\overline{F}} \subset F$, the set-valued mapping Y_j^F is l.h.c. on F^{m+n-1} .

Assumptions C(v) and P(v) follow the solution proposed by Kajii (1988). They avoid the first difficulty. For its part, C(vi) is a direct adaptation of the solution developed by Sun (2006) to solve the second problem. We remark that $\dot{F}$ may depend on $x \in X_i\left(z_{-i}\right)$ and the net $\left(z_{-i}^{\alpha}\right)$. We also note that the net $\left(x^{\alpha}\right)$ converges to x for the norm topology, since it belongs to an affine finite dimensional subspace. Therefore, if X_i is a $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, f\right)$-l.h.c. set-valued mapping, then X_i must be a $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, \tau\right)$–l.h.c. set-valued mapping.

The two remaining problems are related with the fact that even the original economy is supposed to satisfy both the weak survival and the local non-satiation assumptions; this may not be true for the sub-economies. It implies that theorem 2.1 of Bonnisseau (1997) cannot be applied to $\mathcal{E}^F$. We take care of this in section 6 below.

5. The existence result

The assumptions discussed in the preceding sections are precisely those we need to obtain our main result

Theorem 5.1. *Under Assumptions* (C), (P), (B), (BL), (WSA), (R) *and* (PR), *the economy* $\mathcal{E} = \left(\left(X_i, \sum_{i, z_{-i}}, r_i\right)_{i=1}^{m}, \left(Y_j, \varphi_j\right)_{j=1}^{n}, \left(\omega_i\right)_{i=1}^{m}\right)$ *has an equilibrium.*

The above theorem generalizes previous results on the existence of equilibrium with bounded loss pricing rules. Indeed, it extends Theorem 3.1 in the work of Bonnisseau and Meddeb (1999) since it allows the possibility of externalities. It also generalizes the result of Bonnisseau (1997) to an infinite dimensional space.

6. Proof of theorem 5.1

6.1 *Existence of equilibria in the sub-economies*

In this sub-section we apply the existence theorem of Bonnisseau (1997) to a net of finite dimensional sub-economies. We remark that the sub-economy $\mathcal{E}^F$ satisfies Assumptions (P), (B), (BL), (PR), (R) and (C) (except LNS) of theorem 2.1. of Bonnisseau (1997). The lemma 6.1 below shows that, if F is large enough, then each sub-economy also satisfies a weak version of the survival assumption together with the local non-satiation of the preferences on the attainable allocations. Before stating the lemma, we need to introduce two parameters. Let $\tilde{\gamma}$ be a positive real number which satisfies $\tilde{\gamma} > -\sum_{j=1}^{n} \alpha_j$, where α_j is the real number given by Assumption (BL) for each j. Let $\bar{\lambda}$ be a real number such that $\bar{\lambda} \geq \tilde{\gamma}$.

Lemma 6.1. *Under Assumptions* (C), (P), (B), (BL), (WSA), (R) *and* (PR), *there exists a subspace* $\hat{F} \in \mathcal{F}$ *such that for all* $F \in \mathcal{F}$, *if* $\hat{F} \subset F$, *then the sub-economy* $\mathcal{E}^F$ *satisfies*:

(WSA^F): For all $\left(p^F, z^F, \lambda^F\right) \in PE^F \times \left[0, \bar{\lambda}\right]$, if $z^F \in A^F\left(\omega + \lambda^F \chi_M\right)$ then $p^F\left(\sum_{j=1}^{n} y_j^F + \omega + \lambda^F \chi_M\right) > 0$

(LNS^F): For all $\left(\left(x_i^F\right)_{i=1}^{m}, \left(y_j^F\right)_{j=1}^{n}\right) \in A^F(\omega)$, and for all $\varepsilon > 0$, there exists $\left(x_i'^F\right)_{i=1}^{m} \in \prod_{i=1}^{m}\left(X_i^F\left(z_{-i}^F\right) \cap B\left(x_i^F, \varepsilon\right)\right)$, such that $x_i^F \prec^F_{i, z^F_{-i}} x_i'^F$ for all i.

The proof of this lemma is given in the Appendix A. We remark that Assumption (WSA^F) is weaker than Assumption SA in Bonnisseau´s paper. The same applies between (LNS^F) and the local non-satiation assumption of Bonnisseau (1997). Finally, it is also important to note that if $\lambda^F \in \left[0, \bar{\lambda}\right]$ for all $F \in \mathcal{F}$ such that $\hat{F} \subset F$, then $z^F \in A^F\left(\omega + \lambda^F \chi_M\right) \subset A\left(\omega + \bar{\lambda}\chi_M\right)$. This means that the net $\left(z^F\right)_{F \in \mathcal{F}, F \supset \hat{F}}$ belongs to a bounded hence relatively weakly compact set.

We end this sub-section by stating the following result

Lemma 6.2. *Let* $\bar{F}$ *and* $\bar{\bar{F}}$ *be the subspaces from Assumptions* C(v) *and* P(v) *respectively and let* $\hat{F}$ *be the subspace coming from lemma 6.1. Under Assumptions* (C), (P), (B), (BL), (WSA), (R) *and* (PR), *if* $\bar{F} \subset F, \bar{\bar{F}} \subset F$ *and* $\hat{F} \subset F$, *then the sub-economy* $\mathcal{E}^F$ *has an equilibrium* $\left(z^F, p^F\right) \in Z^F \times S^F$.

The proof of this lemma is also given in the Appendix A.

6.2 *From the finite to the infinite dimensional commodity space*

We consider the net $\left(\left(\left(x_i^F\right)_{i=1}^{m}, \left(y_j^F\right)_{j=1}^{n}\right), p^F\right)_{F \in \mathcal{F}}$ of equilibria of the auxiliary economies $\left(\mathcal{E}^F\right)_{F \in \mathcal{F}}$ given by lemma 6.2. From the definition of $\varphi_j^F\left(z^F\right)$, there exist price vectors $\left(\pi_j^F\right)_{j=1}^{n} \in \prod_{j=1}^{n} \varphi_j\left(z^F\right) \subset S^n$ such that $p^F = \pi_{j|F}^F$ for all j. Hence we obtain the net

$\left(\left(x_i^F\right)_{i=1}^m,\ \left(y_j^F\right)_{j=1}^n,\left(\pi_j^F\right)_{j=1}^n\right)_{F\in\mathcal{F}}$. In the following six steps we prove that a limit point exists and it is a bounded loss equilibrium of the economy $\mathcal{E}$.

Step 1. There exists a subnet $\left(\left(x_i^{F(t)}\right)_{i=1}^m,\ \left(y_j^{F(t)}\right)_{j=1}^n,\left(\pi_j^{F(t)}\right)_{j=1}^n\right)_{t\,\in\,(T,\,\geq)}$ which converges to $\left(\left(\bar{x}_i\right)_{i=1}^m,\ \left(\bar{y}_j\right)_{j=1}^n,\left(\bar{\pi}_j\right)_{j=1}^n\right)$ for the product of weak-star topologies. Furthermore, for all *j* and *i*, $\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{t\,\in\,(T,\,\geq)}$ and $\left(\pi_j^{F(t)}\left(x_i^{F(t)}\right)\right)_{t\,\in\,(T,\,\geq)}$ are converging.

Proof

Note that lemma 6.2. implies that $\left(\left(x_i^F\right)_{i=1}^m,\ \left(y_j^F\right)_{j=1}^n\right)_{F\,\in\,\mathcal{F}}$ belongs to $A(\omega)$. Hence, from Assumption (B) it is norm bounded and from Banach-Alaoglu theorem (Ash (1972), p. 162), it remains in a weak-star compact subset of L^{m+n}. The net $\left(\pi_j^F\right)_{F\in\mathcal{F}}$ belongs to *S* which is σ^{ba}-compact. Hence, there exists a subnet $\left(\left(x_i^{F(t)}\right)_{i=1}^m,\ \left(y_j^{F(t)}\right)_{j=1}^n,\left(\pi_j^{F(t)}\right)_{j=1}^n\right)_{t\,\in\,(T,\,\geq)}$ converging to $\left(\left(\bar{x}_i\right)_{i=1}^m,\ \left(\bar{y}_j\right)_{j=1}^n,\left(\bar{\pi}_j\right)_{j=1}^n\right)$ for the product of weak-star topologies. This also implies that the subnets of real numbers $\left(p^{F(t)}\left(y_j^{F(t)}\right)\right)=\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)$ and $\left(p^{F(t)}\left(y_j^{F(t)}\right)\right)=\left(\pi_j^{F(t)}\left(x_i^{F(t)}\right)\right)$ are bounded so that they can be supposed to converge.

Step 2. $\bar{\pi}_1=\bar{\pi}_2=\ldots=\bar{\pi}_n>0$.

Proof

See Bonnisseau (2002). Hereafter, we shall omit the subscript *j* for the sake of simpler notation, i.e., $\bar{\pi}$ $=\bar{\pi}_j$ for all *j*.

Step 3. $\left(\left(\bar{x}_i\right)_{i=1}^m,\ \left(\bar{y}_j\right)_{j=1}^n\right)\in\prod_{i=1}^m X_i\left(\bar{z}_{-i}\right)\times\prod_{j=1}^n Y_j\left(\bar{z}_{-j}\right)$ and $\sum_{i=1}^m \bar{x}_i=\sum_{j=1}^n \bar{y}_j+\omega$

Proof

From step 1, $\left(\left(x_i^{F(t)}\right)_{i=1}^m,\ \left(y_j^{F(t)}\right)_{j=1}^n\right)\in\ Z^{F(t)}\subset\prod_{i=1}^m X_i\left(z_{-i}^{F(t)}\right)\times\prod_{j=1}^n Y_j\left(z_{-j}^{F(t)}\right)$. From Assumptions C(i) and P(i), X_i and Y_j are, for all *i* and *j*, $\left(\prod_{L^{m+n-1}}\sigma^\infty,\sigma^\infty\right)$-closed. Since $\left(z^{F(t)}\right)_{t\,\in\,(T,\,\geq)}$ converges to $\bar{z}$ for the product of weak-star topologies, it follows that

$\overline{z}=\left(\left(\overline{x}_i\right)_{i=1}^m,\ \left(\overline{y}_j\right)_{j=1}^n\right)\in\prod_{i=1}^m X_i\left(\overline{z}_{-i}\right)\times\prod_{j=1}^n Y_j\left(\overline{z}_{-j}\right)$. Since $\sum_{i=1}^m x_i^{F(t)}=\sum_{j=1}^n y_j^{F(t)}+\omega$ for all $t\in T$, one obtains $\sum_{i=1}^m \overline{x}_i=\sum_{j=1}^n \overline{y}_j+\omega$

Step 4. If $x_i \sum_{i,\,\overline{z}_{-i}} \overline{x}_i$ then $\overline{\pi}\left(x_i\right)\geq r_i\left(\overline{\pi}\left(\omega_i\right),\ \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^n\right)$

Proof

Suppose that $\overline{\pi}\left(x_i\right)<r_i\left(\overline{\pi}\left(\omega_i\right),\ \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^n\right)$. Hence, there exists a real number $\varepsilon>0$ such that

$$\overline{\pi}\left(x_i\right)+\varepsilon<r_i\left(\overline{\pi}\left(\omega_i\right),\ \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^n\right) \tag{1}$$

From assumption C(ii), there exists $x_i'\in X_i\left(\overline{z}_{-i}\right)\cap B\left(x_i,\varepsilon/2\right)$ such that $x_i'\succ_{i,\overline{z}_{-i}} x_i$. Hence, since preferences are transitive, $x_i'\succ_{i,\overline{z}_{-i}} \overline{x}_i$. Since the subnet $\left(z_{-i}^{F(t)}\right)_{t\,\in\,(T,\,\geq)}$ converges weakly to $\overline{z}_{-i}$ and X_i is $\left(\prod_{L^{m+n-1}}\sigma^\infty,f\right)$-l.h.c. on L^{m+n-1} (Assumption C(vi)), there exists a finite dimensional subspace $\dot{F}$ such that there is a subnet $\left(x_i'^{F(t)}\right)_{t\,\in\,(T,\,\geq)}$ which converges to x_i', with $\left(x_i'^{F(t)}\right)_{t\,\in\,(T,\,\geq)}\subset x_i'+\dot{F}$ and $x_i'^{F(t)}\ \in X_i\left(z_{-i}^{F(t)}\right)$ for all t. There exists $t_0\in T$ such that $t>t_0$ implies $x_i'+\dot{F}\subset F\left(t\right)$. Hence, $x_i'^{F(t)}\ \in X_i\left(z_{-i}^{F(t)}\right)\cap F\left(t\right)$ for all $t>t_0$.

As $\left(\overline{z}_{-i},x_i',\overline{x}_i\right)\notin\Gamma_i$ and the subnet $\left(x_i^{F(t)}\right)_{t\,\in\,(T,\,\geq)}$ converges to $\overline{x}_i$ for the weak-star topology, there exists $t_1\in T$ such that for all $t>t_1$, $\left(z_{-i}^{F(t)},x_i'^{F(t)},x_i^{F(t)}\right)\notin\Gamma_i$, from Assumption C(iii). Since the subnet $\left(x_i'^{F(t)}\right)$ converges to x_i' for the norm topology, there exists $t_2\in T$ such that for all $t>t_2$, $x_i'^{F(t)}\in B\left(x_i',\varepsilon/2\right)$. Hence, by the completeness of preferences, we deduce that for all $t>\max\left\{t_0,t_1,t_2\right\}$

$$x_i^{F(t)}\in X_i\left(z_{-i}^{F(t)}\right)\bigcap F(t),\ x_i'^{F(t)}\in X_i\left(z_{-i}^{F(t)}\right)\bigcap F(t)\bigcap B\left(x_i',\varepsilon/2\right)\text{ and } x_i'^{F(t)}\succ^{F(t)}_{i,z_{-i}^{F(t)}} x_i^{F(t)}$$

Since the allocation $\left(\left(x_i^{F(t)}\right)_{i=1}^m,\left(y_j^{F(t)}\right)_{j=1}^n,p^{F(t)}\right)$ is an equilibrium of $\mathcal{E}^{F(t)}$, it follows that for all $t>\max\left\{t_0,t_1,t_2\right\}$

$$p^{F(t)}\left(x_i'^{F(t)}\right) > p^{F(t)}\left(x_i^{F(t)}\right) = r_i\left(p^{F(t)}(\omega_i), \left(p^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^{n}\right)$$

As $p^{F(t)} = \pi_{j|F(t)}^{F(t)}$ for all *j*, we have that

$$\pi_j^{F(t)}\left(x_i'^{F(t)}\right) > \pi_j^{F(t)}\left(x_i^{F(t)}\right) = r_i\left(\pi_j^{F(t)}(\omega_i), \left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^{n}\right)$$

We recall that $x_i'^{F(t)} < x_i' + \frac{\varepsilon}{2}\chi_M$. Then, since $\pi_j^{F(t)}$ is a positive linear functional

$$\pi_j^{F(t)}\left(x_i' + \frac{\varepsilon}{2}\chi_M\right) \geq \pi_j^{F(t)}\left(x_i^{F(t)}\right) = r_i\left(\pi_j^{F(t)}(\omega_i), \left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^{n}\right)$$

Passing to the limit, we obtain

$$\overline{\pi}\left(x_i'\right) + \frac{\varepsilon}{2} \geq \lim \pi_j^{F(t)}\left(x_i^{F(t)}\right) = r_i\left(\overline{\pi}(\omega_i), \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^{n}\right)$$

Since $x_i' < x_i + \frac{\varepsilon}{2}\chi_M$ and $\overline{\pi} > 0$, we have

$$\overline{\pi}\left(x_i\right) + \varepsilon \geq r_i\left(\overline{\pi}(\omega_i), \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^{n}\right)$$, which is in contradiction with (1).

Step 5. For all *j*, $\overline{\pi}\left(\overline{y}_j\right) = \lim \pi_j^{F(t)}\left(y_j^{F(t)}\right)$

Proof

Assumption PR(iii) (a) and step 2 imply that $\lim \pi_j^{F(t)}\left(y_j^{F(t)}\right) \geq \overline{\pi}_j\left(\overline{y}_j\right) = \overline{\pi}\left(\overline{y}_j\right)$ for all *j*. From C(iv), for all *i*, one has $r_i\left(\overline{\pi}(\omega_i), \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^{n}\right) \geq r_i\left(\overline{\pi}(\omega_i), \left(\overline{\pi}\left(\overline{y}_j\right)\right)_{j=1}^{n}\right)$, which, together with step 4, implies that

$$\overline{\pi}\left(\overline{x}_i\right) \geq r_i\left(\overline{\pi}(\omega_i),\ \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^{n}\right) \geq r_i\left(\overline{\pi}(\omega_i),\ \left(\overline{\pi}\left(\overline{y}_j\right)\right)_{j=1}^{n}\right) \tag{2}$$

We now prove that $\overline{\pi}\left(\overline{x}_i\right) = r_i\left(\overline{\pi}(\omega_i),\ \left(\overline{\pi}\left(\overline{y}_j\right)\right)_{j=1}^{n}\right)$ for all *i*. Suppose that for some i_0, we have

$$\overline{\pi}\left(\overline{x}_{i_0}\right) > r_{i_0}\left(\overline{\pi}(\omega_{i_0}),\ \left(\overline{\pi}\left(\overline{y}_j\right)\right)_{j=1}^{n}\right)$$

Then, from Assumption C(iv), it follows that $\sum_{i=1}^{m} \overline{\pi}\left(\overline{x}_i\right) > \sum_{j=1}^{n} \overline{\pi}\left(\overline{y}_j\right) + \overline{\pi}(\omega)$. But from step 3 we deduce that $\sum_{i=1}^{m} \overline{\pi}\left(\overline{x}_i\right) = \sum_{j=1}^{n} \overline{\pi}\left(\overline{y}_j\right) + \overline{\pi}(\omega)$, a contradiction.

From (2), it follows that $r_i\left(\overline{\pi}(\omega_i),\ \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^n\right)=r_i\left(\overline{\pi}(\omega_i),\ \left(\overline{\pi}(\overline{y}_j)\right)_{j=1}^n\right)$ for all *i*. From Assumption C(iv) , r_i is strictly increasing in the second variable. Consequently, $\lim\pi_j^{F(t)}\left(y_j^{F(t)}\right)=\overline{\pi}\left(\overline{y}_j\right)$ for all *j*.

From steps 2 and 5 together with Assumption PR (iii) (b), one obtains $\left(\left(\overline{x}_i\right)_{i=1}^m,\ \left(\overline{y}_j\right)_{j=1}^n\right)=\overline{z}\in Z$ and $\overline{\pi}\in\bigcap_{j=1}^n\varphi_j\left(\overline{z}\right)$. From step 3 $\sum_{i=1}^m\overline{x}_i=\sum_{j=1}^n\overline{y}_j+\omega$. Hence, $\left(\overline{z},\overline{\pi}\right)\in PE$ and $\overline{z}\in A(\omega)$. It remains to show that condition a. of definition 2.1. is satisfied.

Step 6. For all *i*, $\overline{x}_i$ is $\succsim_{i,\overline{z}_{-i}}$-maximal in $\left\{x_i\in X_i\left(\overline{z}_{-i}\right):\overline{\pi}\left(x_i\right)\le r_i\left(\overline{\pi}(\omega_i),\ \left(\overline{\pi}\left(\overline{y}_j\right)\right)_{j=1}^n\right)\right\}$

Proof

We have to show that for every agent *i*, if $x_i\succ_{i,\,\overline{z}_{-i}}\overline{x}_i$ then $\overline{\pi}\left(x_i\right)>\overline{\pi}\left(\overline{x}_i\right)$. From step 4, one has $\overline{\pi}\left(x_i\right)\ge\overline{\pi}\left(\overline{x}_i\right)$. Suppose $\overline{\pi}\left(x_i\right)=\overline{\pi}\left(\overline{x}_i\right)$. From steps 4, 5 and Assumption (R) $\overline{\pi}\left(\overline{x}_i\right)=r_i\left(\overline{\pi}(\omega_i),\ \left(\overline{\pi}\left(\overline{y}_j\right)\right)_{j=1}^n\right)>0$. The end of the proof follows directly from Assumption C(i) (compare Bewley (1972)).

■

Remark 6.1. Note that if X_i is simply l.h.c. then $x_i'^{F(t)}\in X_i\left(z_{-i}^{F(t)}\right)$ for all $t\in T$, but $x_i'^{F(t)}$ may not be in $F(t)$. Therefore, in step 4, we could not obtain that $x_i'^{F(t)}\in X_i\left(z_{-i}^{F(t)}\right)\bigcap F(t)$ and $p^{F(t)}\left(x_i'^{F(t)}\right)>p^{F(t)}\left(x_i^{F(t)}\right)$ for all $t>\max\left\{t_0,t_1,t_2\right\}$. Furthermore, there is also no such a finite dimensional subspace $F\in\mathcal{F}$ that $x_i'^{F(t)}\in X_i\left(z_{-i}^{F(t)}\right)\bigcap F(t)$ for all $F(t)$ with $F(t)\supset F$. Then we could not get a $F\in\mathcal{F}$ such that $p^{F(t)}\left(x_i'^{F(t)}\right)>p^{F(t)}\left(x_i^{F(t)}\right)$ for all $F(t)\supset F$.

Remark 6.2. As discussed in sub-section 4.2, Assumptions C(v) and P(v) are used only to get a net of sub-economies each of which has an equilibrium. For its part, Assumption P(vi) is sufficient to assure that the limit of equilibriums is an equilibrium of the whole economy. Therefore, these assumptions are related with mathematical, rather than economic, considerations (See Kajii (1988) and Sun (2006)).

We may consider other sufficient conditions which imply the two needed assumptions and which are easier to understand economically. For the consumption sets, the sufficient condition would be as follows:

For all *i*,

C(v′) For all $z_{-i}\in L^{m+n-1}$, the half line $\left\{t\chi_M:t\ge 0\right\}$ is included in $X_i\left(z_{-i}\right)$

C(vi′) For all $z_{-i} \in L^{m+n-1}$, for all $x_i \in X_i(z_{-i})$, for all $t > 0$, there exists an open neighborhood V of z_{-i} for the product of weak-star topologies such that $x_i + t\chi_M \in X_i(z'_{-i})$ for all $z'_{-i} \in V$.

It should be noted that C(v′) implies that for $z_{-i} \in L^{m+n-1}$, for all $x_i \in X_i(z_{-i})$, for all $t > 0$, $x_i + t\chi_M \in X_i(z_{-i})$ since $X_i(z_{-i})$ is already assumed to be convex and closed. We also note that these assumptions imply that if a net (z^α_{-i}) converges to z_{-i} and $x_i \in X_i(z_{-i})$, then there exists a net $(t^\alpha) \in [0, +\infty)$ which converges to 0 and such that $x_i + t^\alpha \chi_M \in X_i(z^\alpha_{-i})$ for all α. This condition clearly implies the two l.h.c. Assumptions C(v) and C(vi).

For the production set, due to the free-disposal assumption, we have that for $z_{-j} \in L^{m+n-1}$, for all $y_j \in Y_j(z_{-j})$, for all $t > 0$, $y_j - t\chi_M \in Y_j(z_{-j})$. So, it suffices to state:

For all *j*,

P(v′) For all $z_{-j} \in L^{m+n-1}$, for all $y_j \in \partial_\infty Y_j(z_{-j})$, for all $t > 0$, there exists an open neighborhood V of z_{-j} for the product of weak-star topologies such that $y_j - t\chi_M \in Y_j(z'_{-j})$ for all $z'_{-j} \in V$.

As in the case of consumption sets, this assumption implies that if a net (z^α_{-j}) converges to z_{-j} and $y_j \in Y_j(z_{-j})$, then there exists a net $(t^\alpha) \in [0, +\infty)$ which converges to 0 and such that $y_j - t^\alpha \chi_M \in Y_j(z^\alpha_{-j})$ for all α. This condition is clearly stronger than the two l.h.c. Assumptions P(ii) and P(v).

As for the production sets, the above condition is economically interpretable as follows: Since $y_j - t\chi_M$ belongs to the interior of $Y_j(z_{-j})$, it is strictly technologically feasible since we can produce strictly more of all outputs with strictly less of all inputs. So, P(v′) means that a perturbation of the externalities in a well chosen small enough neighborhood V, will not produce a so large effect on the production possibilities such that $y_j - t\chi_M$ would no more be feasible. In some sense, this condition may be understood as the fact that the effect of a small change of the externalities is not too important and can always be counterbalanced by a small move along the half line generated by χ_M. Note that this is stronger than simply assuming that the production sets do not vary too much with respect to small changes in the environment. An analogous interpretation applies for the consumption sets.

Remark 6.3. It is well known that an equilibrium price system in $ba(M, \mathcal{M}, \mu)$ seems like an artificial solution. Hence, Bewley (1972) and others gave sufficient conditions under which equilibrium prices could be chosen from L_1. To obtain a meaningful equilibrium price in our paper, it suffices to adapt the assumptions of Bonnisseau and Meddeb (1999) on the first firm of the economy as follows

(T) Y_1 is a set-valued mapping with convex values, satisfies Assumption (P) and $\varphi_1 = PM_1$.

(PL1) For every $y \in Y_1(z_{-1})$, for every $t > 0$, for every sequence (A_k) of $\mathcal{M}$ such that $\bigcap_{K=1}^{\infty} A_k = \varnothing$, there exists k_0 such that for all $k > k_0$, $y - t\chi_M + \chi_{A_k} \in Y_1(z_{-1})$.

Under these assumptions, if $\left(\left(\left(\bar{x}_i\right)_{i=1}^{m}, \left(\bar{y}_j\right)_{j=1}^{n}\right), \bar{\pi}\right)$ is an equilibrium of the economy $\mathcal{E}$, then $\bar{\pi}$ is an element of $L_1(M, \mathcal{M}, \mu)$. The demonstration is a direct transcription of Bonnisseau and Meddeb´s (1999) proof.

<u>Appendix</u> A

Proof of Lemma 6.1

The proof is inspired from the one of Bonnisseau (2002). Nevertheless it must be adapted to consider externalities and set-valued mappings.

(WSA^F)

We first prove that there exists $\hat{F} \in \mathcal{F}$ such that for all F containing $\hat{F}$, the economy $\mathcal{E}^F$ satisfies Assumption (WSA^F). Suppose, on the contrary, that for all $F \in \mathcal{F}$, there exists $F' \in \mathcal{F}$ such that $F' \supset F$, $\left(p^{F'}, z^{F'}, \lambda^{F'}\right) \in PE^{F'} \times \left[0, \bar{\lambda}\right]$, $z^{F'} \in A\left(\omega + \lambda^{F'}\chi_M\right)$ and $p^{F'}\left(\sum_{j=1}^{n} y_j^{F'} + \omega + \lambda^{F'}\chi_M\right) = 0$. By the definition of $\varphi_j^{F'}$, there exists $\left(\pi_j^{F'}\right) \in \prod_{j=1}^{n} \varphi_j\left(z^{F'}\right)$ such that $\pi_{j|F'}^{F'} = p^{F'}$ for all *j*. Since $z^{F'} \in A\left(\omega + \lambda^{F'}\chi_M\right) \subset A\left(\omega + \bar{\lambda}\chi_M\right)$ for all $F' \in \mathcal{F}$, Assumption (B) implies that $\left(\left(z^{F'}\right), \left(\pi_j^{F'}\right)_{j=1}^{n}, \left(\pi_j^{F'}\left(y_j^{F'}\right)\right)_{j=1}^{n}, \lambda^{F'}\right)_{F' \in \mathcal{F}}$ remains in a compact set for the product of the weak-star topologies and the topology of $\sim^{n+1}$. Hence, there exists a subnet $\left(\left(z^{F'(t)}\right), \left(\pi_j^{F'(t)}\right)_{j=1}^{n}, \left(\pi_j^{F'(t)}\left(y_j^{F'(t)}\right)\right)_{j=1}^{n}, \lambda^{F'(t)}\right)_{t \in (T, \geq)}$ which converges to $\left(\left(\bar{z}\right), \left(\bar{\pi}_j\right)_{j=1}^{n}, \lim\left(\pi_j^{F'(t)}\left(y_j^{F'(t)}\right)\right)_{j=1}^{n}, \lambda\right)$.

Since $z^{F'} \in A\left(\omega + \lambda^{F'}\chi_M\right)$ for all $F' \in \mathcal{F}$, it follows that $\sum_{j=1}^{n} y_j^{F'(t)} + \omega + \lambda^{F'(t)}\chi_M \geq \sum_{i=1}^{m} x_i^{F'(t)} \geq 0$, and since L_+ is weak-star closed, $\sum_{j=1}^{n} \bar{y}_j + \omega + \lambda\chi_M \geq \sum_{i=1}^{m} \bar{x}_i \geq 0$. From Assumptions C(i) and P(i), X_i and Y_j are, for all *i* and *j*, $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, \sigma^{\infty}\right)$-closed. Hence, $\bar{z} \in \prod_{i=1}^{m} X_i\left(\bar{z}_{-i}\right) \times \prod_{j=1}^{n} Y_j\left(\bar{z}_{-j}\right)$.

Using a similar argument than in step 2, one deduces that $\bar{\pi}_j = \bar{\pi} > 0$ for all *j*. Hence, $\sum_{j=1}^{n} \bar{\pi}\left(\bar{y}_j\right) + \bar{\pi}(\omega) + \lambda \geq 0$. Since $p^{F'}\left(\sum_{j=1}^{n} y_j^{F'} + \omega + \lambda^{F'}\chi_M\right) = \pi_j^{F'}\left(\sum_{j=1}^{n} y_j^{F'} + \omega + \lambda^{F'}\chi_M\right) = 0$ for all $F' \in \mathcal{F}$ and all j, we get $\sum_{j=1}^{n} \lim \pi_j^{F'(t)}\left(y_j^{F'(t)}\right) + \bar{\pi}(\omega) + \lambda = 0$. Recalling that Assumption PR (iii) (a) implies that $\lim \pi_j^{F'(t)}\left(y_j^{F'(t)}\right) \geq \bar{\pi}\left(\bar{y}_j\right)$, we deduce

$$0 \leq \sum_{j=1}^{n} \bar{\pi}\left(\bar{y}_j\right) + \bar{\pi}(\omega) + \lambda \leq \sum_{j=1}^{n} \lim \pi_j^{F'(t)}\left(y_j^{F'(t)}\right) + \bar{\pi}(\omega) + \lambda = 0, \text{ hence}$$

$$\sum_{j=1}^{n} \bar{\pi}\left(\bar{y}_j\right) + \bar{\pi}(\omega) + \lambda = 0 \qquad (3)$$

Thus, $\sum_{j=1}^{n} \bar{\pi}\left(\bar{y}_j\right) = \sum_{j=1}^{n} \lim \pi_j^{F'(t)}\left(y_j^{F'(t)}\right)$, which implies that $\bar{\pi}\left(\bar{y}_j\right) = \lim \pi_j^{F'(t)}\left(y_j^{F'(t)}\right)$ for all j. From Assumption PR (iii) (b) it follows that $\bar{z} \in Z$ and $\bar{\pi} \in \bigcap_{j=1}^{n} \varphi_j\left(\bar{z}\right)$ for all j. From Assumption (WSA) one deduces that $\sum_{j=1}^{n} \bar{\pi}\left(\bar{y}_j\right) + \bar{\pi}(\omega) + \lambda > 0$, which is in contradiction with (3)

(LNS^F)

We end the proof of this lemma by showing that there exists $\hat{F} \in \mathcal{F}$ such that the sub-economy $\mathcal{E}^F$ satisfies Assumption (LNS^F) for each F containing $\hat{F}$. First, we prove that preferences are non-satiated on the attainable allocations. Suppose, on the contrary, that for all $F \in \mathcal{F}$, there exists $F' \in \mathcal{F}$, such that $F' \supset F$, $\left(\left(x_i^{F'}\right)_{i=1}^{m}, \left(y_j^{F'}\right)_{j=1}^{n}\right) \in A^{F'}(\omega)$ and for some i_0, it does not exist $\zeta_{i_0}^{F'} \in X_{i_0}^{F'}\left(z_{-i_0}^{F'}\right)$ such that $x_{i_0}^{F} \prec_{i, z_{-i_0}^{F'}}^{F'} \zeta_{i_0}^{F'}$. Since $A^{F'}(\omega) \subset A(\omega)$ for all $F' \in \mathcal{F}$, there exists a subnet $\left(\left(x_i^{F'(t)}\right)_{i=1}^{m}, \left(y_j^{F'(t)}\right)_{j=1}^{n}\right)_{t \in (T, \geq)}$ converging weakly to $\left(\left(\bar{x}_i\right)_{i=1}^{m}, \left(\bar{y}_j\right)_{j=1}^{n}\right)$. From Assumptions C(i) and P(i), X_i and Y_j are, for all i and j, $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, \sigma^{\infty}\right)$-closed. Hence, $\bar{z} \in \prod_{i=1}^{m} X_i\left(\bar{z}_{-i}\right) \times \prod_{j=1}^{n} Y_j\left(\bar{z}_{-j}\right)$. From Assumption C(ii), there exists $\left(\zeta_i\right) \in \prod_{i=1}^{m} X_i\left(\bar{x}_i\right)$ such that $\zeta_i \succ_{i, \bar{z}_{-i}} \bar{x}_i$ for all i. Since the subnet $\left(z_{-i}^{F'(t)}\right)_{t \in (T, \geq)}$ converges weakly to $\bar{z}_{-i}$ and X_i is $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, f\right)$-l.h.c. on L^{m+n-1} (Assumption C(vi)), there exists, for every i, a finite dimensional subspace $\dot{F}_i$ such that there is a subnet $\left(\zeta_i^{F'(t)}\right)_{t \in (T, \geq)}$ which converges to ζ_i, with $\left(\zeta_i^{F'(t)}\right)_{t \in (T, \geq)} \subset \zeta_i + \dot{F}_i$ and $\zeta_i^{F'(t)} \in X_i\left(z_{-i}^{F'(t)}\right)$ for all t. There exists $t_0 \in T$ such that $t > t_0$ implies $\zeta_i + \dot{F}_i \subset F'(t)$ for all i. Hence, $\zeta_i^{F'(t)} \in X_i\left(z_{-i}^{F'(t)}\right) \cap F'(t)$ for all $t > t_0$ and all i.

As $\left(\bar{z}_{-i}, \zeta_i, \bar{x}_i\right) \notin \Gamma_i$ and the net $\left(x_i^{F'(t)}\right)_{t \in (T, \geq)}$ converges to $\bar{x}_i$ for the weak-star topology, there exists t_1 such that for all $t > t_1$, $\left(z_{-i}^{F'(t)}, \zeta_i^{F'(t)}, x_i^{F'(t)}\right) \notin \Gamma_i$, from Assumption C(iii). Hence, for all $t > \max\left\{t_0, t_1\right\}$ and all i, $\zeta_i^{F'(t)}$ and $x_i^{F'(t)}$ belong to $X_i\left(z_{-i}^{F'(t)}\right) \cap F'(t)$ and by the completeness of preferences, $\zeta_i^{F'(t)} \succ_{i, z_{-i}^{F'(t)}}^{F'(t)} x_i^{F'(t)}$. Since $\left\{F'(t) : t \in T\right\} \subset \left\{F' : F' \in \mathcal{F}\right\}$ this contradicts the fact that there exists $F' \in \mathcal{F}$ such that $F' \supset F$, $\left(\left(x_i^{F'} \right)_{i=1}^m, \left(y_j^{F'} \right)_{j=1}^n \right) \in A^{F'}(\omega)$ and for some i_0, it does not exist $\zeta_{i_0}^{F'} \in X_{i_0}^{F'}\left(z_{-i_0}^{F'} \right)$ such that $x_{i_0}^F \prec_{i, z_{-i_0}^{F'}}^{F'} \zeta_{i_0}^{F'}$.

Now, combining the above result with the fact that the convexity of preferences holds true in the sub-economies we deduce that they are locally non-satiated (LNS^F). This ends the proof.

■

Proof of lemma 6.2

Let $F \in \mathcal{F}$ be a finite dimensional subspace such that $F \supset \bar{F}$, $F \supset \bar{\bar{F}}$ and $F \supset \hat{F}$. We start by justifying our choice of the positive real number $\tilde{\gamma}$ as given before Lemma 6.1. To that end, we first note that the proof of Theorem 2.1 of Bonnisseau (1997) starts with an arbitrary real number $\eta > 0$. Then Bonnisseau considers a larger set of price vectors

$$S_\eta = \left\{ p \in \sim^l : \sum_{h=1}^l p_h = 1 \text{ and } p_h \geq -\eta \right\}$$

Bonnisseau also fixes $\bar{\gamma} > 0$ which satisfies

$$\bar{\gamma} > -\sum_{j=1}^n \alpha_j + \max\left\{ p.\left(\sum_{i=1}^m \bar{\xi}_i - \omega \right) : p \in S_\eta \right\}$$

We now proceed to do the same for the sub-economy $\mathcal{E}^F$, recalling that we have replaced the vector *e* by χ_M. To that end, note first that in our case it suffices to fix $\eta = 0$ (compare the proof of theorem 3.1 in Bonnisseau and Meddeb (1999); compare also Bonnisseau and Cornet (1990a)). Consequently (and in contrast to Bonnisseau (1997)), we do not need to consider a set of price vectors larger than S^F. Note also that in our model $\bar{\xi}_i = 0$ for all *i* (compare Bonnisseau (1997)). Hence, we should choose a parameter strictly greater than $-\sum_{j=1}^n \alpha_j + \max\left\{ p^F(-\omega) : p^F \in S^F \right\}$. Evidently, the real number $\tilde{\gamma}$ satisfies the above condition.

We remark that survival assumption in Bonnisseau (1997) is used only to prove lemma 3.2 (b) (p. 224). There, the production plans are close to the attainable allocation in the sense that $\sum_{j=1}^n y_j^F + \omega + \lambda^F \chi_M \geq 0$ and $\lambda^F \in [0, \tilde{\gamma}]$ with the notations of this paper. Since $\tilde{\gamma} \leq \bar{\lambda} < \infty$, we have that condition (WSA^F) of lemma 6.1 is enough to conclude.

For the local non-satiation assumption, we remark that it is used in Bonnisseau (1997) only in claims 5 and 6. In both cases it is applied on the attainable allocations so that condition (LNS^F) of lemma 6.1 is enough to conclude.

■

Appendix B. A particular result

We have shown that our model is sufficiently large to generalize previous works on the existence of general equilibrium. We now prove that Assumption (PR) is satisfied when a firm has a convex production technology.

Theorem B.1

If Y_j is a set-valued mapping with convex values and satisfies Assumption (P), then $\varphi_j = PM_j$ satisfies Assumption (PR).

Proof.

First, one easily checks that PM_j is non-empty and convex valued ((PR) (i)) since for all $z_{-j}^F \in L^{m+n-1}$, $Y_j\left(z_{-j}\right)$ is a convex subset of L and satisfies the free disposal property. We now prove that the correspondence $PM_j^F : Z^F \mapsto S^F$ defined by

$$PM_j^F(z^F) = \left\{p^F \in F^* : \text{there exists } \pi \in PM_j\left(z^F\right) \text{ and } p^F = \pi_{|F}\right\}$$

has a closed graph ((PR) (ii)). Indeed take two sequences $\left(z^n\right)_{n\in\bullet}$ and $\left(p^n\right)_{n\in\bullet}$ of Z^F and S^F respectively, such that $z^n \to \overline{z}$, $p^n \to \overline{p}$ and, for each n, $p^n \in PM_j^F(z^n)$. By the definition of PM_j^F there exists $\pi^n \in PM_j(z^n)$, such that $p^n = \pi_{|F}^n$ for every n. Since π^n belongs to *S*, which is σ^{ba}-compact, there exists a subsequence $\left(\pi^{n_k}\right)_{k\in\bullet}$ converging weakly to $\overline{\pi}$. Since $\pi^{n_k} \in PM_j(z^{n_k})$ we have that $\pi^{n_k}(y_j^{n_k}) \geq \pi^{n_k}(y)$ for all $y \in Y(z_{-j}^{n_k})$. We now prove that $\overline{\pi}(\overline{y}_j) \geq \overline{\pi}(y)$ for all $y \in Y(\overline{z}_{-j})$. Indeed, let $y \in Y_j\left(\overline{z}_{-j}\right)$. Since the subsequence $\left(z_{-j}^{n_k}\right)_{k\in\bullet}$ converges to $\overline{z}_{-j}$, there exists a subsequence of y, $\left(y^{n_k}\right)$, converging to y for the norm topology and $y^{n_k} \in Y_j\left(z_{-j}^{n_k}\right)$ for all *k*, from Assumption P(ii). Since $\pi^{n_k} \in PM_j(z^{n_k})$, one has $\pi^{n_k}\left(y_j^{n_k}\right) \geq \pi^{n_k}\left(y^{n_k}\right)$ for all *k*. At the limit, one obtains $\overline{\pi}\left(\overline{y}_j\right) \geq \overline{\pi}\left(y\right)$. Since this is true for any $y \in Y_j\left(\overline{z}_{-j}\right)$, it follows that $\overline{\pi} \in PM_j(\overline{z})$. Since $\overline{y}_j \in F$ we have that $\overline{p}(\overline{y}_j) = \overline{\pi}(\overline{y}_j) \geq \overline{\pi}(y)$ for all $y \in Y(\overline{z}_{-j})$. Thus, $\overline{p}(\overline{y}_j) \geq \overline{p}(y)$ for all $y \in Y(\overline{z}_{-j}) \cap F$ and, consequently, $\overline{p} \in PM_j^F(\overline{z})$.

Let us prove that the correspondence PM_j satisfies assumption PR(iii). Indeed, let us consider the subnet $\left(z^{F(t)},\ \pi^{F(t)}\right)_{t\in T}$ of the net $\left(z^F,\ \pi^F\right)_{F\in\mathcal{F}} \in Z \times S$, such that

$$\begin{cases} \left(z^{F(t)}, \pi^{F(t)}\right) \to \left(\overline{z},\ \overline{\pi}\right) \text{ for the product of weak-star topologies} \\ \pi^{F(t)} \in PM_j\left(z^{F(t)}\right) \text{ for all } t \in T \\ \pi^{F(t)}\left(y_j^{F(t)}\right) \text{ converges} \end{cases}$$

First, let us prove part (a) of Assumption PR(iii). It suffices to show that $\lim \pi^{F(t)}\left(y_j^{F(t)}\right) \geq \overline{\pi}\left(y\right)$ for all $y \in Y_j\left(\overline{z}_{-j}\right)$. If it is not true, there exists $y' \in Y_j\left(\overline{z}_{-j}\right)$ such that $\lim \pi^{F(t)}\left(y_j^{F(t)}\right) < \overline{\pi}\left(y'\right)$. Hence, there exists ε > 0 such that $\lim \pi^{F(t)}\left(y_j^{F(t)}\right) + \varepsilon < \overline{\pi}\left(y'\right)$. From Assumption P(ii), there exists a subnet $\left(y'^{F(t)}\right)$ which converges to y' for the norm topology and $y'^{F(t)} \in Y_j\left(z_{-j}^{F(t)}\right)$ for all $t \in T$. Since $\pi^{F(t)} \in PM_j\left(z^{F(t)}\right)$ for all $t \in T$, we have that $\pi^{F(t)}\left(y_j^{F(t)}\right) \geq \pi^{F(t)}\left(y'^{F(t)}\right)$ for all $t \in T$.

There exists $t_0 \in T$ such that for all $t > t_0$, $y'^{F(t)} > y' - \varepsilon\chi_M$. Hence, $\pi^{F(t)}\left(y_j^{F(t)}\right) \geq \pi^{F(t)}\left(y'\right) - \varepsilon$ for all $t > t_0$. Passing to the limit, we obtain, $\lim \pi^{F(t)}\left(y_j^{F(t)}\right) \geq \overline{\pi}\left(y'\right) - \varepsilon$, which contradicts the above converse inequality. Consequently, we deduce that $\lim \pi^{F(t)}\left(y_j^{F(t)}\right) \geq \overline{\pi}\left(\overline{y}\right)$.

Finally, let us show that part (b) of Assumption PR(iii) is also satisfied. If we suppose that $\lim \pi^{F(t)}\left(y_j^{F(t)}\right) = \overline{\pi}\left(\overline{y}_j\right)$, then $\overline{\pi}\left(\overline{y}_j\right) \geq \overline{\pi}\left(y\right)$ for all $y \in Y_j\left(\overline{z}_{-j}\right)$ which implies that $\overline{\pi} \in PM_j\left(\overline{z}\right) \subset S$. Together with the fact that $Y_j\left(z_{-j}\right)$ is a convex subset of *L* and satisfies the free disposal property, we have $\overline{y}_j \in \partial_\infty Y_j\left(\overline{z}_{-j}\right)$. ■

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