# Market failures and equilibria in Banach lattices

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**Abstract**

In this paper, we consider an economy with infinitely many commodities and market failures such as increasing returns to scale and external effects or other regarding preferences. The commodity space is a Banach lattice possibly without interior points in the positive cone in order to include most of the relevant commodity spaces in economics. We propose a new definition of the marginal pricing rule through a new tangent cone to the production set at a point of its (non-smooth)-boundary. The major contribution is the unification of many previous works with convex or non-convex production sets, smooth or non-smooth, for the competitive equilibria and for the marginal pricing equilibria, with or without external effects, in finite dimensional spaces as well as in infinite dimensional spaces. In order to prove the existence of a marginal pricing equilibria, we also provide a suitable properness condition on non-convex technologies to deal with the emptiness of the interior of the positive cone. **Mathematics Subject Classification**: 91B50

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## 1 Introduction

General economic equilibrium theory, with infinite dimensional commodity spaces, has been a growing topic during the eighties and nineties and is still a very active field, which allows to incorporate some non-competitive behavior such as asymmetric information. Since [1], the literature has considered infinite dimensional spaces with an order structure to encompass several economically relevant features such as commodity differentiation, uncertainty and infinite time horizon among others. The above extensions are not straightforward extensions of the well documented finite dimensional case. Indeed, closed and order bounded sets are not compact, equilibrium prices are not necessarily continuous and joint continuity does not hold among other difficulties.

The first part of that literature was mainly considering Banach lattices ([2], [3], [4] and [5] among others). The contributions of [6], [7] and [8] have shown that the existence of a general equilibrium whose commodity space is a vector lattice is guaranteed if the price space is also a lattice. More recently, [9] and [10] showed that the lattice structure can be left out, when Walrasian economies are considered.

The above literature only deals with convex production sets and the competitive case. The marginal pricing rule behavior comes from [11] in order to treat the case, when profit-maximization is no longer suitable. For instance, it happens when there are increasing returns to scale or more general types of non-convexities in production. A proper mathematical treatment in a finite dimensional setting was first provided in [12] and then generalized in [13][1] through the Clarke's tangent cone. Indeed, for a production plan $y_j \in Y_j$, a vector $y \in R^L$ belongs to the Clarke's tangent cone if for every sequence $(y_j^n)$ in $Y_j$ converging to $y_j$ and every sequence $(t^n)$ in $]0, \infty[$ such that $t^n \downarrow 0$ there exists a sequence $(y^n)$ in $Y_j$ converging to $y$ such that $y_j^n + t^n y^n \in Y_j$ for all $n \in N$ ([14]). The tangent cone can be geometrically interpreted as a first order approximation of the set $y_j - Y_j$. The advantage of this cone among many others in non-smooth analysis, is that it is always convex and lower semi continuous under a standard free-disposal assumption. The marginal pricing rule is then defined by the fact that prices belong to the Clarke's normal cone, which is the polar of the Clarke's tangent cone. When the commodity space is $R^L$, we can directly define the normal cone as the convex cone generated by the vectors orthogonal to $Y_j$ at $y_j$ and the limits of vectors which are orthogonal to $Y_j$ in the neighborhood of $y_j$. In economic words, the marginal pricing rule means that the $j$-th producer fulfills the first-order necessary condition for profit maximization. The Clarke's normal cone enjoys also the nice property to coincide with the normal half line when the production set is smooth or with the Minkowski normal cone when the production set is convex.

Using the Clarke's normal cone, [15] provides a proof of the existence of a marginal pricing equilibrium when several firms have non-convex technologies. This paper, in turn, has been generalized in two ways by considering externalities and by considering infinitely many commodities. In the first case, the motivation comes from the fact that external factors are, very often, source of non-convexities in both production and consumer preferences. [16] have shown by means of an example that the Clarke's normal cone to the production set for a fixed environment is not longer the right concept since the graph is not closed with respect to the external factor. Consequently, the marginal pricing rule given an external factor must consider not only the close productions for the same level of external effect but also those which are associated to different

[1] The original working paper was published in 1982 and became well known among theorists before its publication in 1990.

but close levels of external effect. This means that the producer takes into account the change of the shape of the production set around a given production, when the external effects vary. The new cone they obtain is larger than the Clarke's one which is actually the cost to be paid to get a normal cone with a closed graph. Some papers, [17], [18] and [19], have studied the marginal pricing equilibria with infinitely many commodities but only in the space $L_\infty$. When externalities are added the same space was used by [20] and [21]. So far, no additional results has been established with non-convexities and/or with externalities in an infinite dimensional setting.

The aim of this paper is first to consider Banach lattice as commodity space to encompass most of the relevant economic models. Second, we provide a suitable definition of the marginal pricing rule based on a tangent cone in Banach lattices for non-convex production sets with external effects. This definition is extending the previous ones in the literature in the sense that if the production sets are convex, the marginal pricing rule coincides with the competitive behavior, if the production sets are smooth like in [22, 20, 21], we recover the standard unique normal price, in a finite dimensional space without external effects, we recover the Clarke's tangent cone, with external effects, the marginal pricing rule introduced in [16], and if the commodity space is $L_\infty$, the concept coincides with the one of [18]. Third, to overcome the difficulty coming from the emptiness of the positive cone, we exhibit a suitable properness condition on productions, which is inspired by the one of [23]. Our main result is to get the existence of a marginal pricing equilibrium under assumptions at the same level of generality than those for the competitive equilibrium.

The motivation for proving existence in such a general framework is not only theoretical but mainly because it allows to cover the relevant economic applications. As pointed out previously, we can encompass many interesting topics like uncertainty, infinite horizon and commodity differentiation among others. Specifically, some of these contexts are (i) endogenous growth models *à la* Romer [24] or [25]. In these models external effects play a key role and the idea of sustained growth requires to consider infinitely many periods. Further, in both models production exhibits increasing returns to scale which is a special case of non-convexity. (ii) Product differentiation and international trade: [26] bases his model on increasing returns and product differentiation and shows the correlation between market size and trade in differentiated products which is relevant with an infinite dimensional commodity space. (iii) Financial markets with bankruptcy chains: a typical form of externality is what can be generated through a "chain" of default values between agents holding "short" and "long" positions, possibly interrelated mutually. In these context, models of general equilibrium with incomplete markets with default or bankruptcy and infinitely many states of the world fit as special cases of our model (see, for example, the special issue of Journal of Mathematical Economics, 1996).

From a mathematical point of view, the definition of the new tangent cone combines the tools of the Clarke's tangent cone extending by the one in [16] to encompass the externalities and the one of [18] for the infinite dimensional. But to overcome the fact that the open ball may not be order bounded, we have to combine weak* open neighborhoods and order intervals to get the desired continuity properties. Another contribution of the paper is to provide a so-called local star-shaped assumption on the production sets to avoid an abstract condition (Assumption TC) borrowed from [18]. This assumption is satisfied in finite dimensional spaces thanks to the free disposal assumption or when production sets are convex. It translates the idea that the shape of the production set is not too chaotic like a fractal. It has the advantage of being directly set on the production sets and not as a property of the tangent cone.

As for the existence proof, we first consider the case where the interior of the positive cone is nonempty. Then we use the method of [27] by considering the limit of a sequence of equilibria in suitable truncated economies with finite dimensional commodity spaces as in [18]. To consider the case where the positive cone has an empty interior, we first restrict the commodity space to the principal ideal generated by the vector $e$. Then a new topology is generated on the restricted space which is stronger than the relative (norm) topology, so that the interior of the positive cone is non-empty. We get the existence of an equilibrium with this restricted commodity space by using the first step. To go beyond, a key properness assumption is required to prove that equilibrium in the restricted space is actually an equilibrium in the original economy. More precisely, the question at stake is to extend the price functional from the principal ideal to the whole space as it is the case for competitive equilibrium but also to check that the necessary assumptions are inherited when we restrict the commodity space to the order ideal. That is why we adapt the properness condition of [23] to our setting. Nevertheless, we need a uniform properness instead of a pointwise properness since the definition of the normal cone involves not only the reference production but also productions in a neighborhood. To the best of our knowledge, the properness condition in [28] is the only one involving non-convex productions sets. It is suitable for supporting weakly Pareto optimal allocations but not for the more demanding existence result.

The paper is organized as follows: Section 2 deals with the mathematical structure. Section 3 presents the model and the Assumptions together with the specification of the new marginal pricing rule. In Section 4, we state an existence result, when the interior of the positive cone has a non-empty interior. Then, in Section 5, we extend the analysis to more general Banach lattices whose positive cones have an empty interior. For that purpose, we state our properness condition. Several technical proofs of propositions, lemmas and theorems are given in Appendix.

## 2 Terminology and Notation

Let $L$ be a Banach lattice, i.e., a Riesz space equipped with a complete lattice norm denoted by $\|\cdot\|$. The space $L$ is also endowed with a Hausdorff locally convex-solid topology $\tau$, which is weaker than the norm topology and such that all order intervals are $\tau$-compact[2]. $L_+ = \{x \in L : x \geq 0\}$ is the positive cone of $L$ which is $\tau$-closed.

Let $L^*$ be the topological dual of $L$. For $x \in L$ and $\pi \in L^*$, $\pi(x)$ is the evaluation or the value of the commodity bundle $x$ for the price $\pi$. We denote by $\sigma^*$ the weak* topology on $L^*$ and by $\| \cdot \|^*$ the dual norm. Let $L^M$ be the product space given by the cartesian product of $M$ copies of the space $L$. If each space $L$ is endowed with the topology $\tau$, we denote by $\prod_{L^M} \tau$ the product topology of $L^M$. The product space $L^M$ is also a Banach lattice. For all $x \in L$, ${}_\tau(x)$ is the set of $\tau$-neighborhood of $x$ and for all $z \in L^M$, $\prod_{L^M} {}_\tau(z)$ is the set of $\prod_{L^M} \tau$-neighborhood of $z$.

Let $A : L^M \mapsto L$ be a correspondence. We say that $A$ has $\tau$-closed values if for every $x \in A$, $A(x)$ is a $\tau$-closed subset of $L$.

For further details on infinite dimensional spaces and Banach lattices, we refer to [31] and to [30].

# 3 The Model

There are finite sets of consumers and producers $I$ and $J$ respectively. Each element $z = ((x_i)_{i \in I}, (y_j)_{j \in J})$ belongs to $L^{I+J}$, where $L^{I+J}$ is the product space given by the Cartesian product of $\#I + \#J$ copies of the space $L$. Each consumer $i$ has a *consumption set* and a other regarding *preference relation*, which depends upon the actions of the other economic agents. Formally, for each $i \in I$, $X_i : L^{I+J} L_+$ is the consumption correspondence. For the environment $z \in L^{I+J}$, $X_i(z) \subset L_+$ is the consumption set of the $i$-th consumer. We denote by $\succeq_{i,z}$ the binary preference relation of agent $i$ on the set $X_i(z)$. This relation is assumed to be complete, reflexive and transitive. The relation of strict preference $x \succ_{i,z} x'$ is then defined by $x \succeq_{i,z} x'$ and not $x' \succeq_{i,z} x$. We do not assume that we can compare two commodity bundles if they do not share the same environment. Let $\omega_i \in L_+$ be the initial endowment of the $i$-th agent such that $\omega_i \in X_i(z)$ for all $z \in L^{I+J}$. Let us denote the total initial endowment of the economy by $\omega = \sum_{i \in I} \omega_i \neq 0$.

Each producer $j$ has a production set which also depends upon the actions of the other agents. For each $j \in J$, $Y_j : L^{I+J} L$ is the production correspondence. For the environment $z \in L^{I+J}$, $Y_j(z) \subset L$ is the set of all feasible production plans for the $j$-th producer. We denote the $\| \cdot \|$-boundary of $Y_j(z)$ by $\partial Y_j(z)$[3].

[2]The norm is a lattice norm if for all $(x, \xi) \in L^2$, $|x| \leq |\xi|$ implies $\|x\| \leq \|\xi\|$ where $|x|$ is the absolute value of $x$, that is $x \vee 0 + (-x) \vee 0$. $B(0,1) = \{x \in L : \|x\| < 1\}$ denotes the open ball of center 0 and radius 1, $B(0, \varepsilon) = \varepsilon B(0,1)$ the open ball of center 0 and radius $\varepsilon$, $B(x, \varepsilon) = x + \varepsilon B(0,1)$ the open ball of center $x$ and radius $\varepsilon$, $\bar{B}(0,1) = \{x \in L : \|x\| \leq 1\}$ the closed ball of center 0 and radius 1. A subset $E$ of $L$ is called solid if for all $(x, y) \in E \times L$, if $|y| \leq |x|$ then $y \in E$. For $x, y \in L$ with $x \leq y$, the interval $[x, y]$ is defined by the set $\{z \in L : x \leq z \leq y\}$ . The principal ideal generated by $x \in L$ is $L(x) = \bigcup_{n \in} n[-x, x]$ which is a vector sublattice of $L$. An element $x \in L_+$ is an order unit if $L(x) = L$. The $\| \cdot \|$-topology is locally solid with a base of neighborhoods of zero which are radial and circled sets.

[3]We are considering productions on the boundary of the production set since under the free-disposal assumption when $\int L_+ \neq \emptyset$, $\partial Y_j(z) = \{y \in L : (\{y\} + \int L_+) \cap Y_j(z) = \emptyset\}$

The price set is given by $S = \{\pi \in L_+^* : \|\pi\|_{L^*} = 1\}$.

Let $r_i : {}^{1+J} \to$ be the *wealth function* of the $i$-th consumer. If $\pi \in S$ and $(y_j)_{j\in J} \in \prod_{j\in J} Y_j(z)$, her/his wealth is $r_i(\pi(\omega_i), (\pi(y_j))_{j\in J})$. This encompasses the private ownership economy as a particular case, i.e., when $r_i(\pi(\omega_i), (\pi(y_j))_{j\in J}) = \pi(\omega_i) + \sum_{j\in J} \theta_{ij}\pi(y_j)$ for $\theta_{ij} \geq 0$ and $\sum_{i\in I} \theta_{ij} = 1$ for all $j \in J$ and $i \in I$.

For a given environment $z \in L^{I+J}$ and a given initial endowments $\omega' \in L_+$, we denote by $A(\omega', z)$ the set of attainable productions, that is,

$$A(\omega', z) := \{(y'_j) \in \prod_{j\in J} \partial Y_j(z) : \sum_{j\in J} y'_j + \omega' \in L_+\}$$

In order to consider only consistent situations, we define the set

$$Z := \{z = ((x_i)_{i\in I}, (y_j)_{j\in J}) \in L^{I+J} : \forall i \in I,\ x_i \in X_i(z);\ \forall j \in J,\ y_j \in \partial Y_j(z)\}$$

The set of *weakly efficient attainable allocations* corresponding to a given total initial endowment $\omega' \in L_+$ is given by

$$A(\omega') := \{z = ((x_i)_{i\in I}, (y_j)_{j\in J}) \in Z : \sum_{i\in I} x_i = \sum_{j\in J} y_j + \omega'\}$$

Finally, we introduce $e \in L_+$ as a reference commodity bundle satisfying $\|e\| = 1$. A natural candidate is $e = \omega/\|\omega\|$ but this is not always the right choice. Indeed, when $L$ is the space $L_\infty$ of essentially bounded measurable functions, it is convenient to choose $e = \chi$ the constant function equal to 1.

In the following, we consider two cases namely when the interior of $L_+$ is nonempty or when it is empty. In the first case, we assume that $e$ belongs to the interior of $L_+$. In that case, since 0 belongs to the interior of the order interval $[-e, e]$, then the norm $\|\cdot\|$ is equivalent to the lattice norm associated to $e$ defined by:

$$\|x\|_e = \inf\{t \in_+ : |x| \leq te\}$$

So, we will then assume that the norm $\|\cdot\|$ is actually the norm $\|\cdot\|_e$, which implies that the closed balls are actually order intervals.

When the interior of $L_+$ is empty, we assume that $e$ is a quasi-interior point of $L_+$ (i.e. $e$ is strictly positive), which means that the principal ideal $L(e)$ generated by $e$ is norm-dense in $L_+$.

## 3.1 Marginal Pricing Equilibrium

The economic motivation for the study of the marginal pricing equilibrium comes from the second theorem of welfare economics telling us that at a Pareto optimal allocation, there exists a common price satisfying the marginal pricing rule for each producer. A producer follows the marginal pricing rule with respect to a given price at a given production if it maximizes the profit not necessarily on the whole production set, as it is the case when the production set is convex, but on a first order approximation, which is formally defined as the tangent cone at the given production. This result holds true in very general frameworks (see, e.g., [12], [29], [32] and [18]) where the marginal pricing rule is defined by a notion of normal cone, the set of outward direction, which is the dual of the tangent cone, the set of inward or quasi-inward directions.

But, when we depart from the convex production sets where the Minkowski normal cone is the only relevant one, there are many possibilities, many normal cones, to define the marginal pricing rule, specially in infinite dimensional spaces. It is known since [33] that some of these normal cones are compatible with the existence of an equilibrium and some other no. This is mainly due to two key properties, convexity and closedness of the graph. But, on the other hand, the smaller is the normal cone, the more informative is the existence result. So, we have to combine these two opposite criteria in the search of the right definition of the marginal pricing rule.

In finite dimensional space, the Clarke's normal cone as introduced in [13] and extended in [16] to encompass externalities is the right concept and, in some sense, the smallest one. Unfortunately, its graph is not closed for the relevant topologies in infinite dimensional commodity spaces. The existence proof requires the closedness for the weak and weak-star topologies, whereas the Clarke's normal cone is based only on the norm topology. That is why we borrow from [18] an adaptation of the definition of the Clarke's normal cone to introduce weak open neighborhood. But, since these neighborhoods for the weak topologies are very large, actually always unbounded, we make this transposition into two steps in order to get a smaller normal cone. Actually, we first consider a bounded neighborhood of the relevant parameters represented below by the parameter $\rho$ and then we take the intersection for all non-negative $\rho$.

Furthermore, in Riesz spaces, we can also use the order structure and order intervals to define order convergences. But, we have to distinguish between the case where the positive cone has a nonempty interior or not. Indeed, when the interior is nonempty, the norm closed unit ball is an order interval and so there is no need to consider order intervals. But, in this paper where we deal with general Riesz spaces, we also introduce the order interval $r[-e, e]$, in the definition below to measure the "distance" to the production set, which is tighter than the ball $B(0, re)$. So, the definition of the tangent cone is as follows:

For every $(\bar{y}_j, z) \in \partial Y_j(z) \times Z \subset L^{1+I+J}$ and $\rho > 0$ we let:

$$\hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j) := \left\{ \nu \in L : \begin{array}{c} \exists\ \eta > 0 \text{ such that } \forall\ r > 0, \\ \exists\ V \in_{\prod_{L^{I+J}} \tau} (z),\ U \in_{\tau} (\bar{y}_j)\ \textit{and}\ \varepsilon > 0 \\ \forall\ z' \in B(z, \rho) \cap V, \\ \forall\ \bar{y}'_j \in B(\bar{y}_j, \rho) \cap U \cap Y_j(z') \\ \text{and } \forall\ t \in ]0, \varepsilon[,\ \exists\ \xi \in r[-e,\ e] \text{ such that} \\ \bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) \in Y_j(z') \end{array} \right\}$$

Then, we define the set

$$\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) := \cap_{\rho > 0} \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j).$$

By polarity, we define

$$\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) = [\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)]^o = \{\pi \in L^* : \pi(\nu) \leq 0 \ \forall \nu \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)\}.$$

To comment this definition, we remark that the introduction of the neighborhood $V$ for the parameter $z$ is the way to take into account the external effects on production. Then, the intersection over the parameter $\rho$ and the perturbation of $\nu$ by the additional term $\eta(y_j - y'_j)$ is a way to get the conditional closeness of the graph with respect to the topology $\tau$. Finally, choosing $\xi$ in the order interval $r[-e, e]$ is the way to narrow the range of $\xi$, or, in other words, to say that the vector $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j))$ is close to the set $Y_j(z')$ for the order relation which is tighter than the norm. Actually, if $e$ belongs to the interior of $L_+$, then we can replace the order interval by a ball, which is closer to the standard definition of a tangent cone in the spirit of Clarke [14]. Finally, note that the set $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ is not necessarily convex.

From an economic point of view, the idea is very simply: for computing prices, the owner of the $j$-th firm observes not only the current production plan $y_j$ given the environment $z$, but also all production plans $y'_j$ close to $y_j$ that are consistent with the environments $z'$ which are close to $z$. This is the same idea in the models of [34], [35] and [18]. The main difference is the mathematical notion of *nearness* that we use.

The *marginal pricing rule* is formally defined as: given $(\bar{y}_j, z) \in \partial Y_j(z) \times Z$, the $j$-producer chooses the prices in $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \cap S$. In words, to compute the prices according to the marginal pricing rule, the producer takes into account the fact that his production set depends on external effects. Finally, the set of *production equilibria* is $PE := \{(\pi, z) \in S \times Z : \pi \in \cap_{j \in J} \hat{\mathcal{N}}_{Y_j(z)}(y_j)\}$

We are now able to state the definition of a marginal pricing equilibrium.

**Definition 1** *A marginal pricing equilibrium of the economy $\mathcal{E}$ is an element $(z = (x_i)_{i \in I}, (y_j)_{j \in J}), \pi)$ in $Z \times S$ such that:*

1. $\pi(x_i) \leq r_i(\pi(\omega_i), (\pi(y_j))_{j \in J})$ *and* $\pi(x'_i) > r_i(\pi(\omega_i), (\pi(y_j))_{j \in J})$ *whenever* $x'_i \succ_{i,z} x_i$ *for all* $i \in I$.

2. $\pi \in \hat{\mathcal{N}}_{Y_j(z)}(y_j) \cap S$ *for all* $j \in J$

3. $\sum_{i \in I} x_i = \sum_{j \in J} y_j + \omega$

## 3.2 Basic Assumptions

We now posit the following assumptions. Some of them became standard in the literature with increasing returns

**Assumption (C)**
For all $i \in I$

1. $X_i$ is a convex-valued correspondence with a $(\prod_{L^{I+J}} \tau, \tau)$ closed graph. Furthermore, for all $z \in L^{I+J}$, $0 \in X_i(z)$ and $X_i(z)$ is a solid subset of $L_+$.

2. For all $z \in L^{I+J}$, the half-line $\{\delta e : \delta > 0\}$ is included in $X_i(z)$. For all $x'_i \in X_i(z)$ and for all $\delta > 0$ there exists a neighborhood $V \in_{\prod_{L^M} \tau} (z)$ such that $x'_i + \delta e \in X_i(z')$ for all $z' \in V$.

3. For every $z \in L^{I+J}$, and all $\bar{x}_i \in X_i(z)$ both sets $\{x'_i \in X_i(z) : \ x'_i \succeq_{i,z} \bar{x}_i\}$ and $\{x'_i \in X_i(z) : \ \bar{x}_i \succeq_{i,z} x'_i\}$ are $\| \cdot \|$-closed. For all $x'_i \in X_i(z)$ such that $x'_i \succ_{i,z} x_i$, for all $t \in ]0,1[$, $tx'_i + (1-t)x_i \succ_{i,z} x_i$. For every $z \in A(\omega)$, there exists $(x'_i)_{i \in I} \in \prod_{i \in I}(X_i(z) \cap L(e))$ such that $x'_i \succ_{i,z} x_i$ for all $i \in I$.

4. The set $G_i = \{(x'_i, \bar{x}_i, z) \in L^2 \times L^{I+J} : (x'_i, \bar{x}_i) \in X_i(z)^2, x'_i \succeq_{i,z} \bar{x}_i\}$ is a $(\| \cdot \| \times \tau \times \prod_{L^{I+J}} \tau)$-closed subset of $L^2 \times L^{I+J}$.

5. The wealth function $r_i : R^{1+J} \to R$ is continuous and increasing. Furthermore, for all $((v_i), (v_j)) \in R^{I+J}$, $\sum_{i \in I} r_i(v_i, (v_j)_{j \in J}) = \sum_{i \in I} v_i + \sum_{j \in J} v_j$ and if $\sum_{i \in I} r_i(v_i, (v_j)_{j \in J}) > 0$ then $r_i(v_i, (v_j)_{j \in J}) > 0$ for all $i$.

**Assumption (P)** For every $j \in J$

1. $Y_j : L^{I+J} L$ has a $(\prod_{L^{I+J}} \tau, \tau)$-closed graph.

2. For every $z \in L^{I+J}$, $Y_j(z) \cap L_+ = \{0\}$ and $Y_j$ satisfies the free-disposal condition, that is, $Y_j(z) - L_+ = Y_j(z)$.

3. For all $z \in L^{I+J}$, for all $\bar{y}_j \in \partial Y_j(z)$ and for all $\delta > 0$, there exists $V \in_{\prod_{L^M} \tau} (z)$, such that $\bar{y}_j - \delta e \in Y_j(z')$ for all $z' \in V$.

**Assumption B (Boundedness)** For all $\omega' \geq \omega$
there exists $b \in_+$ such that, for all $z \in Z$, $A(\omega', z) \cap L(e)^J \subset [-be, be]^J$.

**Assumption SA (Survival)** For all $z \in Z$, for all $t \in_+$ and for all $(\pi, (\bar{y}_j)) \in S \times A(\omega + te, z)$, if $\pi \in \cap_{j \in J} \hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$, then $\pi(\sum_{j \in J} \bar{y}_j + \omega + te) > 0$

**Assumption TC (Tangent cone)** For all $z \in Z$, for all $j$ and for all $\bar{y}_j \in \partial Y_j(z)$, $0 \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$.

**Remark 1**. Note that we assume non-satiation only on the attainable sets. Together with the convexity of preferences, we have local non-satiation. We assume continuity of preference relations with respect to the external factors and to the consumption bundles.

**Remark 2**. Assumption C(1) states that $X_i(z)$ is solid (i.e., it contains order-intervals). We do so because we are considering consumption sets which are not equal to the positive cone $L_+$ (See [4]). We make use of this property in Section 5.

**Remark 3**. By C(1), $X_i(z)$ is a closed and convex subset of $L_+$. Hence, by C(2), one gets that for all $x \in X_i(z)$, $x + \delta e \in X_i(z)$ for all $\delta \geq 0$. We note that Assumption C(2) also implies that if a net $(z^\gamma)$ $\prod_{L^{I+J}} \tau$-converges to $z$, then for every $x'_i \in X_i(z)$ there exists a sequence $(t^\gamma) \in [0, \infty[$ which converges to 0 and such that $x'_i + t^\gamma e \in X_i(z^\gamma)$ for all $\gamma$. This is stronger than the standard lower hemi-continuity condition on $X_i$, which will be exploited in Section 4.

From an economic point of view, since $x'_i + \delta e$ is a possible consumption, Assumption C(2) means that a perturbation of the externalities in a well chosen small neighborhood $V$ will not produce a so large effect on the consumption possibilities as to $x'_i + \delta e$ would be no longer possible. Thus a small change in the externalities has a relatively small impact in consumption and can be counterbalanced by a small move along the half-line generated by $e$.

**Remark 4**. Assumption C(3) says that feasible consumption vectors are non-satiated in $L(e)$ rather than in $L$. However, as we shall see later, $L(e)$ includes the relevant commodity bundles in the economy. We refer to [8] who states an analogous assumption for a competitive economy.

**Remark 5**. With regard to the production sets, Assumption P(3) implies that if a net $(z^\gamma)$ $\prod_{L^{I+J}} \tau$-converges to $z$, then for every $y'_j \in Y_j(z)$ there exists a sequence $(t^\gamma) \in [0, \infty[$ which converges to 0 and such that $y'_j - t^\gamma e \in Y_j(z^\gamma)$ for all $\gamma$. As in Remark 3, we point out that this is stronger than the standard lower hemi-continuity condition. Nevertheless, in finite dimensional spaces, P(3) is satisfied if $z \mapsto Y_j(z)$ is lower hemi-continuous under the free disposal assumption.

This assumption is economically interpretable similarly to the one on the consumption side. We point out that, in [20], there have been posited less strong assumptions than C(3) and P(3) although they were less economically meaningful. Besides the economic interpretation there is a technical reason for Assumption P(3). We need it in order to prove that profit maximization is the behavior followed by producers when production correspondences are convex-valued (See Proof of Proposition 3 (6)).

**Remark 6**. Boundedness assumption says that feasible production vectors belonging to a principal ideal are order bounded which, in turn, implies that the attainable set in $L(e)^{I+J}$ belongs to $[-\omega' - \#Jbe, \omega' + \#Jbe]$.

For spaces whose positive cone has a non-empty interior, Assumption B is automatically satisfied if $A(\omega')$ is norm-bounded.

**Remark 7** As consequences of Assumption TC, we have $-L_+ \subset \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ and $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \subset L^*_+$. Later we will show that $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$ is not reduced to the null vector. This Assumption is also made in [18].

The following assumption is stronger than Assumption TC but it exhibits two advantages: first, it is stated directly on the production sets, not on the tangent cone, and it has an easier economic interpretation as a reinforcement of Assumption P(3).

**Assumption SP(3)** For all $z \in L^{I+J}$, for all $\bar{y}_j \in \partial Y_j(z)$ and for all $\delta > 0$ there exists $V \in_{\prod_{L^{I+J}} \tau} (z)$, $U \in_\tau (\bar{y}_j)$ such that for all $z' \in V$, for all $\bar{y}'_j \in Y_j(z') \cap U$, for all $t \in ]0,1]$, $t(\bar{y}_j - \delta e) + (1-t)\bar{y}'_j \in Y_j(z')$.

If $Y_j(\cdot)$ satisfies Assumption SP(3), we say that $Y_j$ is $\prod_{L^{I+J}} \tau$*-locally star-shaped with respect to* $\bar{y}_j - \delta e$ whenever $\bar{y}_j \in Y_j(z)$. We easily check that Assumption SP(3) is stronger than Assumption P(3). If $Y_j(z)$ is convex for all $z$, Assumption SP(3) is clearly a consequence of Assumption P(3). Now we prove that Assumption TC is satisfied if $Y_j$ satisfies Assumptions P and SP(3).

Let $z \in Z$ and $\bar{y}_j \in \partial Y_j(z)$. Let $\rho > 0$ and take $\eta = 1$. Let $r > 0$ and choose $\delta > 0$ smaller than $r$. Let $V$ and $U$ be the weak-open sets as given by Assumption SP(3) and $\varepsilon = 1$. Then, for all $z' \in B(z, \rho) \cap V$, for all $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U \cap Y_j(z')$ and for all $t \in ]0, \varepsilon[$, we have that the vector $t(\bar{y}_j - \delta e) + (1-t)\bar{y}'_j = \bar{y}'_j + t(\bar{y}_j - \bar{y}'_j - \delta e) \in Y_j(z')$, which means $0 \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$. Since this is true for all $\rho$, we conclude that $0 \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$.

We finally remark that Assumption SP(3) is a consequence of Assumption P(3) when $L$ is finite dimensional. So, Assumption SP(3) is necessary only to deal with infinite dimensional spaces in the non-convex case.

If Assumption P is satisfied and $L$ is finite dimensional, let $z \in L^{I+J}$ and $\bar{y}_j \in \partial_{Y_j(z)}(\bar{y}_j)$. Let $\delta > 0$ and $V$ be the neighborhood of $z$ coming from Assumption P(3) that is $\bar{y}_j - \delta e \in Y_j(z')$ for all $z' \in V$. Let us consider the neighborhood of $\bar{y}_j$, $U = \{\bar{y}_j - \delta e\} + \int L_+$. From the free-disposal assumption and since $L$ is finite dimensional, the norm topology is equal to the weak topology. Thus, for all $z' \in V$, for all $\bar{y}'_j \in Y_j(z') \cap U$ and for all $t \in ]0, 1[$, the vector $t(\bar{y}_j - \delta e) + (1-t)\bar{y}'_j = \bar{y}'_j - t(\bar{y}'_j - \bar{y}_j + \delta e)$ belongs to $Y_j(z')$ from the free-disposal assumption and the fact that $\bar{y}'_j - \bar{y}_j + \delta e$ belongs to $L_+$.

### 3.3 Comments on the Marginal Pricing Rule

Proposition 3 below allows us to compare the marginal pricing rule with the existing ones in the literature. We first define two correspondences in order to compare with the existing notions in finite dimensional commodity spaces. For every $(\bar{y}_j, z) \in \partial Y_j(z) \times Z \subset L^{1+I+J}$, we let

$$\hat{T}_{Y_j(z)}(\bar{y}_j) := \left\{ \nu \in L : \begin{array}{c} \forall r > 0 \ \exists \ \varepsilon > 0 \ : \forall z' \in B(z, \varepsilon) \\ \forall \bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z'), \forall t \in ]0, \varepsilon[ \\ \exists \ \xi \in B(0, r) \ : \bar{y}'_j + t(\nu + \xi) \in Y_j(z') \end{array} \right\}$$

and

$$\hat{N}_{Y_j(z)}(\bar{y}_j) = [\hat{T}_{Y_j(z)}(\bar{y}_j)]^o = \{\pi \in L^* : \pi(\nu) \leq 0 \ \forall \nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)\}$$

**Proposition 2** *Suppose that Assumption* P *holds, then for every* $(\bar{y}_j, z)$ *in* $\partial Y_j(z) \times Z$

1. $\hat{T}_{Y_j(z)}(\bar{y}_j)$ *is a convex cone and* $-L_+ \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$.

2. $\hat{T}_{Y_j(z)}(\bar{y}_j) \subset T_{Y_j(z)}(\bar{y}_j)$ *where the former is the Clarke's tangent cone to the set* $\partial Y_j(z)$ *at the vector* $\bar{y}_j$. *Without externalities, both notions coincide.*

The proof is given in Appendix. The inclusion for the tangent cones implies the reverse inclusion for the normal cones: $N_{Y_j(z)}(\bar{y}_j) \subset \hat{N}_{Y_j(z)}(\bar{y}_j) \subset L_+$, where $N_{Y_j(z)}(\bar{y}_j)$ is the Clarke's normal cone.

We provide below some properties of the marginal pricing rule. The most important feature are the following: when the production sets $Y_j(z)$ are convex, then the producers maximize their profit taken the price as given as well as the environment $z$; when there is no externality and the interior of the positive cone is nonempty, it coincides with the concept of [18]; if $L$ is finite dimensional, then we recover the concept of [16] and if, furthermore, we have no externality, we are back to the definition of the marginal pricing rule based on the Clarke's normal cone as introduced by Cornet in [13]. Since the proof is long, we defer it to the Appendix.

**Proposition 3** *Under Assumptions* P *and* TC*, for* $(\bar{y}_j, z) \in \partial Y_j(z) \times Z$*.*

1. $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ *is a cone,* $-L_+ \subset \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ *and* $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \subset L^*_+$ *is a convex and* $\sigma^*$*-closed cone.*
2. $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$ *and* $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \supset \hat{N}_{Y_j(z)}(\bar{y}_j)$*.*
3. $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) \subset \mathcal{T}_{Y_j(z)}(\bar{y}_j)$ *where the former is the small tangent cone of* [18] *to the set* $Y_j(z)$ *at* $\bar{y}_j$*.*
4. *If* $e \in \int L_+$ *then* $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ *is a convex cone.*
5. *If* $L$ *is finite dimensional, then* $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \cap S = MP(\bar{y}_j, z)$*, where the later is the marginal pricing rule in finite dimensional economies of* [16]*.*
6. *If* $Y_j$ *is convex-valued*

$$\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) = \{\pi \in L^* : \pi(\bar{y}_j) \geq \pi(y'_j) \text{ for all } y'_j \in Y_j(z)\}$$

**Remark** Let us note that since for all $z \in L^{I+J}$ the set $Y_j(z)$ satisfies free disposal, then $\nu \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ if and only if for all $\rho > 0$ there exists $\eta > 0$ such that for all $r > 0$ there are $V \in_{\prod_{L^{I+J}} \tau} (z)$, $U \in_\tau (y_j)$ and $\varepsilon > 0$ such that for all $z' \in B(z, \rho) \cap V$, for all $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U \cap Y_j(z')$ and for all $t \in ]0, \varepsilon[$ it follows that $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) - re) \in Y_j(z')$.

In order to complete our comparison with the literature, we consider now smooth production sets, i.e., when for all $z \in Z$ the technology is described by $Y_j(z) = \{\zeta_j \in L : f_j(\zeta_j, z) \leq 0\}$ where $f_j : L \times L^{I+J} \to R$ is a differentiable mapping called the transformation function. Let us consider the following assumption:

**Assumption SB** (Smooth boundary)

1. $f_j$ is continuous for the $\tau \times \prod_{L^{I+J}} \tau$ topology on $L \times L^{I+J}$;
2. $f_j(\cdot, z)$ is Fréchet differentiable and Lipschitz on $L$;
3. $\nabla_1 f_j(\zeta_j, z)$ belongs to $L^*_+ \setminus \{0\}$ if $\zeta_j \in \partial Y_j(z)$ where $\nabla_1 f_j(\zeta_j, z)$ is the gradient of $f_j$ with respect to $\zeta_j$;
4. $\nabla_1 f_j$ is continuous for $\tau \times \prod_{L^{I+J}} \tau$ toppology on $L \times L^{I+J}$ and the norm topology on $L^*$.

**Lemma 4** *If the production set $Y_j(\cdot)$ is described by a transformation function $f_j$ satisfying Assumption* SB*, then $\hat{\mathcal{N}}_{Y_j(z)}(\zeta_j)$ is the half-line generated by $\nabla_1 f_j(\zeta_j, z)$, which is also the Clarke's Normal cone to $Y_j(z)$ at $\zeta_j$.*

The proof of this lemma is given in Appendix.

# 4 When $L_+$ Has a Nonempty Interior

In this section, we assume that the positive cone $L_+$ has a nonempty interior[4] and the vector $e \in intL_+$. Then $[-e,\ e]$ is a neighborhood of 0 ([30]). At this stage, we show the link between the closure of the graph of $\hat{N}_{Y_j(\cdot)}(\cdot)$ with $\hat{\mathcal{N}}_{Y_j(\cdot)}(\cdot)$. It will become a key result for the proof of existence of equilibria when $intL_+ \neq \emptyset$

**Proposition 5** *Let $(z^\gamma, \pi^\gamma)_{\gamma\in\Gamma}$ be a net in $A(\omega) \times S$ converging to $(z, \pi)$ for the product-topology $\prod_{L^{I+J}} \tau \times \sigma^*$ such that*

*a) $\pi^\gamma(y_j^\gamma)$ converges for all $j$*

*b) $\pi^\gamma \in \hat{N}_{Y_j(z^\gamma)}(y_j^\gamma)$ for all $\gamma \in \Gamma$.*

*Then, $\pi(y_j) \leq \lim_\gamma \pi^\gamma(y_j^\gamma)$ and $\pi \in \hat{\mathcal{N}}_{Y_j(z)}(y_j) \cap S$ if $\pi(y_j) = \lim_\gamma \pi^\gamma(y_j^\gamma)$*

We first state and prove the following lemma.

**Lemma 6** *Let $\rho > 0$ and $\nu \in \hat{\mathcal{T}}^\rho_{Y_j(z)}(\bar{y}_j)$. From the definition of $\hat{\mathcal{T}}^\rho_{Y_j(z)}(\bar{y}_j)$, there exists $\eta > 0$ such that for $\bar{r} > 0$, there exist $\varepsilon_{\bar{r}} > 0$, $V_{\bar{r}}$ and $U_{\bar{r}}$ such that for all $z' \in B(z, \rho) \cap V_{\bar{r}}$ and all $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U_{\bar{r}} \cap Y_j(z')$, $t \in ]0, \varepsilon_{\bar{r}}[$, $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) - \bar{r}e) \in Y_j(z')$. Then, for all $z' \in B(z, \rho) \cap V_{\bar{r}}$ and all $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U_{\bar{r}} \cap Y_j(z')$, $\nu + \eta(\bar{y}_j - \bar{y}'_j) - 2\bar{r}e \in \hat{T}_{Y_j(z')}(\bar{y}'_j)$.*

**Proof.** Let $\rho > 0$ arbitrary and let $\nu \in \hat{\mathcal{T}}^\rho_{Y_j(z)}(\bar{y}_j)$. Let $z' \in B(z, \rho) \cap V_{\bar{r}}$ and let $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U_{\bar{r}} \cap Y_j(z')$. Let $\varepsilon' > 0$ smaller than $\varepsilon_{\bar{r}}$ and $\frac{\bar{r}}{\eta}$ and such that $B(z', \varepsilon') \subset B(z, \rho) \cap V_{\bar{r}}$ and $B(\bar{y}'_j, \varepsilon') \subset B(\bar{y}_j, \rho) \cap U_{\bar{r}}$. Hence, for all $z'' \in B(z', \varepsilon')$, for all $\bar{y}''_j \in B(\bar{y}'_j, \varepsilon') \cap Y_j(z'')$ and for all $t \in ]0, \varepsilon'[$, we have that $\bar{y}''_j + t(\nu + \eta(\bar{y}_j - \bar{y}''_j) - \bar{r}e) = \bar{y}''_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \eta(\bar{y}'_j - \bar{y}''_j) - \bar{r}e) \in Y_j(z'')$. Since $\bar{y}''_j \in B(\bar{y}'_j, \varepsilon')$, $\eta(\bar{y}'_j - \bar{y}''_j) \geq -\eta\varepsilon' e \geq -\bar{r}e$. Hence, by the free-disposal property, $\bar{y}''_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) - 2\bar{r}e) \in Y_j(z'')$. So, for all $r > 0$, the vector $\nu + \eta(\bar{y}_j - \bar{y}'_j) - 2\bar{r}e$ satisfies the definition of $\hat{T}_{Y_j(z')}(\bar{y}'_j)$ with $\epsilon'$ has above and $\xi = 0 \in B(0, r)$. ■

Now we proceed with the proof of Proposition 5:

From Assumption B, $A(\omega)$ is bounded, so, there exists $\rho > 0$ large enough such that for all $(z', z'') \in A(\omega)^2$, $z'' \in B(z', \rho)$ and $y''_j \in B(y'_j, \rho)$ for all $j$.

Let $\nu \in \hat{\mathcal{T}}^\rho_{Y_j(z)}(y_j)$. From the previous Lemma, there exists $\eta > 0$ such that for $\bar{r} > 0$, there exist $V_{\bar{r}}$ and $U_{\bar{r}}$ such that for all $z' \in B(z, \rho) \cap V_{\bar{r}}$ and all $\bar{y}'_j \in B(y_j, \rho) \cap U_{\bar{r}} \cap Y_j(z')$, $\nu + \eta(y_j - \bar{y}'_j) - 2\bar{r}e \in \hat{T}_{Y_j(z')}(\bar{y}'_j)$.

[4]Examples of this kind of spaces are $\mathcal{L}_\infty$ and the space $C(K)$ of real-valued continuous functionals on a compact Hausdorff space $K$ endowed with the supremum norm

Since $(z^\gamma, \pi^\gamma)_{\gamma\in\Gamma}$ converges to $(z,\pi)$ for the product-topology $\prod_{L^{I+J}} \tau \times \sigma^*$ and $\pi^\gamma(y_j^\gamma)$ converges for every $j$ and $\pi^\gamma \in \hat{N}_{Y_j(z^\gamma)}(y_j^\gamma)$ for every $\gamma \in \Gamma$, there exists $\gamma_0$ such that for every $\gamma \geq \gamma_0$, it follows $z^\gamma \in V_{\bar{r}} \cap A(\omega) \subset V_{\bar{r}} \cap B(z,\rho)$ and $y_j^\gamma \in U_{\bar{r}} \cap B(y_j,\rho) \cap Y_j(z^\gamma)$. It follows from the above lemma that for all $\gamma \in \Gamma$, $\nu + \eta(y_j - y_j^\gamma) - 2\bar{r}e \in \hat{T}_{Y_j(z^\gamma)}(y^{\gamma_j})$. Since $\pi^\gamma \in \hat{N}_{Y_j(z^\gamma)}(y_j^\gamma)$, we get $\pi^\gamma(\nu + \eta(y_j - y_j^\gamma) - 2\bar{r}e) \leq 0$. Hence, $\pi(\nu) + \eta\pi(y_j) - \eta \lim_\gamma \pi^\gamma(y_j^\gamma) \leq 2\bar{r}$. Since this is true for all $\bar{r} > 0$ we deduce that $\pi(\nu) + \pi(y_j) \leq \lim_\gamma \pi^\gamma(y_j^\gamma)$. From Assumption TC, $0 \in \hat{\mathcal{T}}_{Y_j(z)}(y_j) \subset \hat{\mathcal{T}}^\rho_{Y_j(z)}(y_j)$. Consequently, the previous inequality leads to $\pi(y_j) \leq \lim_\gamma \pi^\gamma(y_j^\gamma)$.

If $\pi(y_j) = \lim_\gamma \pi^\gamma(y_j^\gamma)$, then we get $\pi(\nu) \leq 0$ for all $\nu \in \hat{\mathcal{T}}^\rho_{Y_j(z)}(y_j)$, which is larger than $\hat{\mathcal{T}}_{Y_j(z)}(y_j)$ and we conclude that $\pi \in \hat{\mathcal{N}}_{Y_j(z)}(y_j)$.

To get the existence of a marginal pricing equilibrium in the present section, we follow the approach of Bewley [27]. Thus, we construct a directed set of finite dimensional economies from which we can apply Theorem 3.1 of [16] in each of them in order to get a net of equilibria in the truncated economies, which converges to an equilibrium of the original economy.

Let $F$ be a finite dimensional subspace of $L$ containing the vectors $e$ and $(\omega_i)_{i\in I}$. We denote by $\mathcal{F}$ the family of such subspaces directed by set inclusion. For every $F \in \mathcal{F}$, we define its positive cone by $F_+ = F \cap L_+$ and its interior is $\int F_+ = F \cap \int L_+$. We endow each $F$ with an euclidean structure, the associated norm being denoted $\|\cdot\|_F$ such that $\|e\|_F = 1$ and $\{e^{\perp_F}\} \cap F_+ = \{0\}$, where $e^{\perp_F}$ denotes the orthogonal space to $e$. The inner product of $F$ will be denoted by $\langle \cdot,\cdot \rangle_F$. Hence, the dual space of $F$ is $F$ itself, i.e., $p^F(x) = \langle p^F, x\rangle_F$. For $x \in F$ we denote by $B^F(x,r) = B(x,r) \cap F$ (resp. $\bar{B}^F(x,r)$) the open (resp. closed) ball of center $x$ and radius $r$ in $F$ for the euclidean structure.

The truncated consumption correspondence for the commodity space $F$ is given by $X_i^F : F^{I+J} F_+$ such that $X_i^F(z^F) = X_i(z^F) \cap F_+$. In the same way, the truncated production correspondence $Y_j^F : F^{I+J} F$, is defined by $Y_j^F(z^F) = Y_j(z^F) \cap F$. Consequently, we define the set

$$Z^F = \{z^F \in F^{I+J} : \forall i \in I, x_i^F \in X_i^F(z^F) \text{ and } \forall j \in J, y_j^F \in \partial Y_j^F(z^F)\}$$

and let us note that $\partial Y_j^F(z^F) \subset \partial Y_j(z^F) \cap F$ thanks to the free-disposal assumption.

Let $S^F = \{p^F \in F_+^0 : \langle p^F, e\rangle_F = 1\}$, where $F_+^0$ denotes the positive polar cone of $F_+$. The revenue of the $i$-th consumer in the truncated economy is given by the same revenue function $r_i$. The restricted preference relation on $X_i^F(z^F)$ is $\succeq^F_{i,z^F}$. Therefore, subeconomies are fully described by

$$\mathcal{E}^F = \{(X_i^F, \succeq^F_{i,z^F}, r_i, \omega_i)_{i\in I}, (Y_j^F)_{j\in J}\}$$

for all $F \in \mathcal{F}$.

We point out that for all $F \in \mathcal{F}$, for all $z^F \in F^{I+J}$, and for all $i$ and $j$, $X_i^F(z^F)$ and $Y_j^F(z^F)$ are non-empty subsets of $F_+$ and $F$ respectively. Proposition 3 guarantees that for all $F \in \mathcal{F}$ and for all $(\bar{y}_j, z) \in F^{1+I+J}$, the set $\hat{\mathcal{N}}_{Y_j^F(z)}(\bar{y}_j) \cap S^F = MP^F(\bar{y}_j, z) =$

$$conv \left\{ p \in F : \begin{array}{c} \exists \ (\bar{y}_j^n, z^n) \in L \times L^{I+J} \text{ and } (p^n) \in S^F \\ \text{such that } (\bar{y}_j^n, z^n) \to (\bar{y}_j, z), (p^n) \to p \\ \bar{y}_j^n \in \partial Y_j(z^n) \text{ and } p^n \in N_{Y_j(z^n)}(\bar{y}_j^n) \end{array} \right\}$$

Finally we let

$$A^F(\omega') = \left\{ z^F \in Z^F : \sum_{i \in I} x_i^F = \sum_{j \in J} y_j^F + \omega' \right\} \subset A(\omega')$$

and

$$PE^F = \left\{ (p^F, z^F) \in S^F \times A^F(\omega) : p^F \in \bigcap_{j \in J} MP_j(y_j^F, z^F) \right\}$$

## 4.1 Existence of marginal pricing equilibria

We can state now the following existence result

**Theorem 7** *The economy* $\mathcal{E} = \{(X_i, \succeq_i, r_i, \omega_i)_{i \in I}, (Y_j)_{j \in J}\}$ *has a marginal pricing equilibrium if it satisfies Assumptions* (C)*,* (P)*,* (B)*,* (SA) *and* (TC)*.*

Within the literature concerning externalities and increasing returns, the above result extends the one of [16] by allowing an infinite dimensional space. It also generalizes [18] since external effects are taken into account. In addition, this model encompasses the one of [21] since we are considering non-smooth technologies. With regard to the papers in which a general pricing rule is considered (for instance, [19] and [20]), we note that even though a larger number of situations can be considered, the existence results require two additional assumptions about boundedness of the losses of the firms and continuity of the pricing rules.

## 4.2 Proof of Theorem 7

The proof follows the guidelines of [21]. The difference relies on the fact that now we are considering non-smooth technologies.

*Marginal pricing equilibria with a finite dimensional commodity space*

We note that the Survival and the Local Non-Satiation assumption may not be satisfied in the economy $\mathcal{E}^F$. Consequently, it seems that Theorem 3.1 of [16] cannot be applied. However, Lemma 8 below shows that whether finite dimensional commodity subspaces are large enough, then weaker versions of those assumptions hold thanks to Proposition 5. Then, Proposition 9 will show that these are, however, sufficient for proving an equilibrium in such subeconomies. We first prepare the ground for stating this lemma. Let $\bar{\vartheta} > 0$ be a real number. By Assumption B, there exists $a > 2\bar{\vartheta}$, such that for all $z \in L^{I+J}$, $A(\omega + \bar{\vartheta}e, z) \subset [-\frac{a}{2}e, \ \frac{a}{2}e]^J$ and $A(\omega + \bar{\vartheta}e) \subset [-\frac{a}{2}e, \ \frac{a}{2}e]^{I+J}$. Let $\bar{r} > 2a$ such that $\{\omega + \bar{\vartheta} e\} + [-\#Jae,\ \#Jae] \subset [-\bar{r}e,\ \bar{r}e]$. Let $\bar{\lambda}$ be a real number such that $\bar{\lambda} \geq 4\#J\bar{r} + \|\omega\|_F$. We will show later that the parameter $\bar{\lambda}$ is large enough so that all relevant productions $(y_j)$ belongs to $A^F(\omega + \bar{\lambda} e, z^F)$ whatever is the environment $z^F$.

**Lemma 8** *Under Assumptions* (C), (P), (B), (SA) *and* (TC), *there exists a subspace $\hat{F} \in \mathcal{F}$ such that for all $F \in \mathcal{F}$, if $\hat{F} \subset F$, then the subeconomy $\mathcal{E}^F$ satisfies:*

*($SA^F$): For all $(p^F, z^F, \lambda) \in PE^F \times \left[0, \bar{\lambda}\right]$, if $(y_j^F)_{j\in J} \in A^F(\omega + \lambda e, z^F)$ then $\langle p^F, \sum_{j\in J} y_j^F + \omega + \lambda e\rangle_F > 0$*

*($LNS^F$): For all $((x_i^F)_{i\in I}, (y_j^F)_{j\in J}) \in A^F(\omega)$, there exists $(x'_i)_{i\in I} \in \prod_{i\in I} X_i^F(z^F)$ such that $x'_i \succ_{i,z^F} x_i^F$ for all $i \in I$.*

The proof is in Appendix

**Proposition 9** *Let $\hat{F} \in \mathcal{F}$ be the subspace coming from Lemma 8. Under Assumptions* (C), (P), (B), (SA) *and* (TC) *if we have $\hat{F} \subset F$ then the subeconomy $\mathcal{E}^F$ has an equilibrium $(z^F, p^F) \in Z^F \times S^F$.*

**Proof** We remark that the economy $\mathcal{E}^F$ does not satisfy Assumptions (SA) and (LNS) but only (SA$^F$) and (LNS$^F$) so we need to explain how we can nevertheless apply the existence result of [16].

The non-satiation assumption (LNS) is only used in [16] for attainable allocations, so (LNS$^F$) is sufficient.

In [16], the Survival Assumption is used in Lemma 4.2 (3), Lemma 4.4 and in Claim 4.3. For Lemma 4.2 (3) and Claim 4.3, the Survival Assumption is applied only for productions plans which satisfy $\sum_{j\in J} y_j + \omega + \vartheta e \geq 0$ with $\vartheta \leq \bar{\vartheta}$. So Assumption (SA$^F$) is sufficient since $\bar{\lambda} > \bar{\vartheta}$.

As for Lemma 4.4, we need to introduce the following notations:

Let $D^F := \bar{B}^F(0, \bar{\lambda})^I \times \bar{B}^F(0, \bar{r})^I$ and $Z_D^F := Z^F \cap D^F$

$\lambda_j^F : e^{\perp_F} \times F^{I+J} \to$

$\qquad (s_j, z) \mapsto \lambda_j^F(s_j, z)$

$\Lambda_j^F(s_j, z) = s_j - \lambda_j^F(s_j, z)e \in \partial Y_j^F(z)$

$\lambda_0^F : e^{\perp_F} \times F^{I+J} \to$

$\qquad (s_j, z) \mapsto \lambda_0^F(s_j, z)$

$\Lambda_0^F(s_j, z) = s_j - \lambda_0^F(s_j, z)e \in \partial(-F_+)$

$\theta^F((s_j)_{j\in J}, z) = \sum_{j\in J} \lambda_j^F(s_j, z) + \lambda_0^F(\sum_{j\in J} s_j - proj_{e^{\perp_F}}\omega, z) - \langle \omega, e\rangle_F,$

$$\Delta^F((s_j)_{j\in J}, z) = \left\{ (p_j - p)_{j\in J} \;\middle|\; \begin{array}{l} p_j \in MP_j(\Lambda_j(s_j, z), j \in J \\ p \in N_{-F_+}(\Lambda_0^F(s_0, z)) \cap S^F \end{array} \right\}$$

$GM_{\bar{\vartheta},\alpha}^F = \left\{((s_j)_{j\in J}, z) \in (e^{\perp_F})^J \times Z_D^F \mid \bar{\vartheta} \leq \theta^F((s_j)_{j\in J}, z) \leq \alpha\right\}$

$\alpha = \max\left\{\theta^F((s_j)_{j\in J}, z) \mid ((s_j)_{j\in J}, z) \in (\bar{B}^F(0, 2a) \cap \{e^{\perp_F}\})^J \times Z_D^F\right\}$

Then, we remark that, in the proof of Lemma 4.4, Assumption (SA) is only necessary to prove that $0 \notin \Delta^F((s_j)_{j\in J}, z)$ for all $((s_j)_{j\in J}, z) \in GM_{\bar{\vartheta},\alpha}^F$. This assumption is applied to productions $(\Lambda_j^F(s_j,z))$ for some $((s_j)_{j\in J},z)\in GM^F_{\bar\vartheta,\alpha}$. But Lemma 4.3 in [16] shows that $\sum_{j\in J}\Lambda_j^F(s_j,z)+\omega+\alpha e\geq 0$ since we have that $\theta^F((s_j)_{j\in J},z)\leq\alpha$. So, to prove that Assumption (SA$^F$) is sufficient, it remains to prove that $\alpha\leq 4\#J\bar r+\|\omega\|^F<\bar\lambda$.

For $\Lambda_j^F(s_j,z)=s_j-\lambda_j^F(s_j,z)e\in\partial Y_j^F(z)$, we claim that, $\|\lambda_j^F(s_j,z)|\leq\|s_j\|$ for all $j\in J$. Indeed, if $|\lambda_j^F(s_j,z)|>\|s_j\|$ then, if $\lambda_j^F(s_j,z)>0$, $s_j<\lambda_j^F(s_j,z)e$. There exists $\varepsilon>0$ such that $s_j+\varepsilon e<\lambda_j^F(s_j,z)e$. Let $\xi\in B^F(s_j-\lambda_j^F(s_j,z)e,\varepsilon)$, hence $\xi<s_j-\lambda_j^F(s_j,z)e+\varepsilon e<0$. Since $0\in\partial Y_j(z)$, it implies that $\Lambda_j^F(s_j,z)$ belongs to $Y_j(z)-int(F_+)$ which contradicts $\Lambda_j^F(s_j,z)\in\partial Y_j^F(z)$. Now, let us consider $\lambda_j^F(s_j,z)<0$, then $\lambda_j^F(s_j,z)e<s_j$. There exists $\varepsilon'>0$ such that $\lambda_j^F(s_j,z)e<s_j-\varepsilon' e$. Consequently, for all $\xi'\in B^F(s_j-\lambda_j^F(s_j,z)e,\varepsilon')$, it follows $\xi'>s_j-\lambda_j^F(s_j,z)-\varepsilon' e>0$. Hence, $\Lambda_j^F(s_j,z)\in int(F_+)$ which is a contradiction with the fact that $\Lambda_j^F(s_j,z)\in\partial Y_j^F(z)$ and the claim is proved. Consequently, $\|\Lambda_j^F(s_j,z)\|\leq 2\|s_j\|$. In an analogous manner, $\Lambda_0^F(u,z)\in\partial(-F_+)$ implies $|\lambda_0^F(u,z)|\leq\|u\|$. Let $((s_j),z)\in(B(0,2a)\cap\{e^{\perp_F}\})^J\times Z^D$. From the previous results and the fact that $proj_{e^{\perp_F}}\omega\leq\|\omega\|$, it follows that

$$|\theta^F((s_j)_{j\in J},z)|\leq 8\#Ja+\|\omega\|<4\#J\bar r+\|\omega\|<\bar\lambda$$

which ends the proof of the proposition.

*The limit argument*

We have a net of finite dimensional equilibria $\mathcal{E}^F=((x_i^F)_{i\in I},(y_j^F)_{j\in J},p^F)$ for every $F\in\mathcal{F}$. Note that $z^F\in A(\omega)$. The next lemma shows that we can extend the price vector $p^F$ which corresponds to the marginal pricing rule in $\mathcal{E}^F$ to a continuous linear functional on the entire space $L$ which is also a marginal pricing rule.

**Lemma 10** *Let $Y_j$ be a production correspondence satisfying Assumption* P. *For all $F\in\mathcal{F}$, for all $(\bar y_j,z)\in\partial Y_j^F(z)\times Z^F$ and for all $p\in MP_j(\bar y_j,z)$, there exists $\pi\in\hat N_{Y_j(z)}(\bar y_j)\cap S$ such that $\pi_{|F}$ is colinear to $p$.*

**Proof** First of all we show that $\text{int}\hat T_{Y_j(z)}(\bar y_j)\cap F\subset\text{int}\hat T_{Y_j^F(z)}(\bar y_j)$. Unlike [18], we cannot use hypertangency since $\hat T_{Y_j(z)}(\bar y_j)$ is not the Clarke's tangent cone. Let $\nu\in\text{int}\hat T_{Y_j(z)}(y_j)\cap F$, hence there is $\delta>0$ such that $B(\nu,\delta)\subset\hat T_{Y_j(z)}(\bar y_j)$. Take $\nu'\in B(\nu,\delta)\cap F$. Since $\nu'\in\hat T_{Y_j(z)}(\bar y_j)\cap F$, for every $r>0$ there exists $\varepsilon>0$ associated to $\frac{r}{2}$ such that for all $z'\in B(z,\varepsilon)\cap F^{I+J}$, for all $\bar y_j'\in B(\bar y_j,\varepsilon)\cap F\cap Y_j(z')$ and for all $t\in]0,\varepsilon[$ there exists $\xi\in L$ such that $\|\xi\|<\frac{r}{2}$ and $\bar y_j'+t(\nu'+\xi)\in Y_j(z')$. By free-disposal $\bar y_j'+t(\nu'-\frac{r}{2}e)\in Y_j(z')$. Since $\nu'$ and $\frac{r}{2}e$ belong to $F$, one has that $\bar y_j'+t(\nu'-\frac{r}{2}e)\in Y_j(z')\cap F$ and thus $\nu'\in\hat T_{Y_j^F(z)}(\bar y_j)$, which implies that $\nu\in\int\hat T_{Y_j^F(z)}(\bar y_j)$.

Let us consider the linear manifold $N=\{\nu\in F:p(\nu)=0\}$. It is clear that $N\cap\int\hat T_{Y_j^F(z)}(\bar y_j)=\emptyset$, whence $N\cap\int\hat T_{Y_j(z)}(\bar y_j)=\emptyset$. By Hahn-Banach extension theorem there exists a closed manifold $H = \{\nu \in L : \hat{\pi}(\nu) = 0\}$ containing $N$ and not intersecting $\int \hat{T}_{Y_j(z)}(\bar{y}_j)$. Clearly, $\hat{\pi}$ is continuous, colinear to $p$ and it belongs to $\hat{N}_{Y_j(z)}(\bar{y}_j)$.

The conclusion of this lemma provides a price in $\hat{N}_{Y_j(z)}(\bar{y}_j)$, which is smaller than $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$. Nevertheless, the forthcoming limit argument shows that the equilibrium price is actually in $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$ since the cone $\hat{N}_{Y_j(z)}(\bar{y}_j)$ does not satisfy the necessary closedness property.

For all $j \in J$, let $\pi_j^F$ be an extension of $p^F$ in $\hat{N}_{Y_j(z)}(y_j^F)$. The net

$$(z^F, (\pi_j^F)_{j \in J})_{F \in \mathcal{F}} \in A(\omega) \times \prod_{j \in J} \hat{N}_{Y_j(z)}(y_j^F) \cap S$$

has a limit point. Indeed, by Assumption (B), $(z^F)_{F \in \mathcal{F}}$ belongs to a $\prod_{L^{I+J}} \tau$-compact set while $(\pi_j^F)_{j \in J}$ belongs to a $\prod_{L^{J,*}} \sigma^*$-compact set from Alaoglu's Theorem. Hence, there exists a subnet $(z^{F(\gamma)}, (\pi_j^{F(\gamma)})_{j \in J})_{\gamma \in \Gamma}$ which converges to $(z, (\pi_j)_{j \in J})$.

On the other side, since the nets $\left(\langle p^{F(\gamma)}, y_j^{F(\gamma)} \rangle\right)_{\gamma \in \Gamma} = \left(\pi_j^{F(\gamma)}(y_j^{F(\gamma)})\right)_{\gamma \in \Gamma}$ and $\left(\langle p^{F(\gamma)}, x_i^{F(\gamma)} \rangle\right)_{\gamma \in \Gamma} = \left(\pi_j^{F(\gamma)}\left(x_i^{F(\gamma)}\right)\right)_{\gamma \in \Gamma}$ are bounded, we can assume without any loss of generality that they converge in $^J$ and $^I$.

We now prove that there is a marginal pricing equilibrium of the economy $\mathcal{E}$.

- *Claim 1.* $\pi_1 = \pi_2 = \ldots = \pi_J = \pi > 0$

  **Proof** Let $x \in L$. There exists $F \in \mathcal{F}$ such that $x \in F$. There exists $\gamma_0 \in \Gamma$ such that $\gamma > \gamma_0$ implies $F \subset F(\gamma)$. As for all $j \in J$, $\pi^{F(\gamma)}_{j|F(\gamma)} = p^{F(\gamma)}$, for $\gamma > \gamma_0$ we obtain $\pi^{F(\gamma)}_{j|F(\gamma)}(x) = p^{F(\gamma)}(x)$ for all $j \in J$. Taking limits we deduce $\pi_1(x) = \pi_2(x) = \ldots = \pi_J(x)$. Since $x$ is an arbitrary vector in $L$ we get $\pi_1 = \pi_2 = \ldots = \pi_J$. Consequently, we can say that $\pi_j = \bar{\pi}$ for all $j \in J$

  On the other hand, $e \in intL_+$ and $p^{F(\gamma)}(e) = \pi_j^{F(\gamma)}(e) = 1$ for all $\gamma \in \Gamma$. Hence, $\pi_j(e) = \pi(e) > 0$ and the claim is proved.

- *Claim 2.* $z \in \prod_{i \in I} X_i(z) \times \prod_{j \in J} Y_j(z)$ and $\sum_{i \in I} x_i = \sum_{j \in J} y_j + \omega$

  **Proof** This is a consequence of Assumptions C(1) and P(1)

- *Claim 3.* If $x_i' \succ_{i,z} x_i$, then $\pi(x_i') \geq r_i\left(\pi(\omega_i), \lim_\gamma(\pi_j^{F(\gamma)}(y_j^{F(\gamma)})_{j \in J})\right)$ for all $i \in I$

  **Proof** There exists a subspace $F_0$ such that $x_i' \in F_0$. There also exists $\gamma_0$ and a positive sequence $\lambda^{F(\gamma)}$ in $[0, \infty[$ converging to 0 from above such that for all $\gamma > \gamma_0$, $F_0 \subset F(\gamma)$ and $x_i' + \delta^{F(\gamma)} e \in X_i(z^{F(\gamma)})$ by Assumption

C(2), whence $x'_i + \delta^{F(\gamma)}e \in X_i(z^{F(\gamma)}) \cap F(\gamma)$. Since $(x_i, x'_i, z) \notin G_i$ (see Assumption C(4)), then there exists $\gamma_1$ such that for all $\gamma$ larger than $\gamma_0$ and $\gamma_1$, $(x_i^{F(\gamma)}, x'_i + \delta^{F(\gamma)}e, z^F(\gamma)) \notin G_i$ and thus $x'_i + \delta^{F(\gamma)}e \succ_{i,z^{F(\gamma)}} x_i^{F(\gamma)}$. By the equilibrium conditions in the economy $\mathcal{E}^{F(\gamma)}$ and the fact that $\pi_{j|F(\gamma)}^{F(\gamma)} = p^{F(\gamma)}$, $\pi_j^{F(\gamma)}(x'_i + \delta^{F(\gamma)}e) > r_i(\pi_j^{F(\gamma)}(\omega), (\pi_j^{F(\gamma)}(y_j^{F(\gamma)})_{j\in J})$. Since $x'_i + \delta^{F(\gamma)}e$ converges to $x'_i$ in the norm-topology and $\pi_j = \pi$ for all $j$, we have

$$\pi(x'_i) \geq r_i(\bar{\pi}(\omega), lim_t(\pi_j^{F(t)}(y_j^{F(t)})_{j\in J}))$$

and the claim is proved.

- *Claim 4.* $z \in A(\omega)$ and $\pi \in \bigcap_{j\in J} \hat{\mathcal{N}}_{Y_j(z)}(y_j) \cap S$

  **Proof** By Proposition 5, $\lim \pi_j^{F(\gamma)}(y_j^{F(\gamma)}) \geq \pi_j(y_j)$ for all $j \in J$. By the above result and the local non satiation Assumption C(3), we deduce that $\pi(x_i) = r_i\left(\pi(\omega_i), (\pi(y_j))_{j\in J}\right)$ for all $i \in I$ which, in turn, using the market clearing condition, implies that $\lim \pi_j^{F(\gamma)}(y_j^{F(\gamma)}) = \pi(y_j)$. Hence again by Proposition 5, $z \in A(\omega)$ and $\pi \in \bigcap_{j\in J} \hat{\mathcal{N}}_{Y_j(z)}(y_j) \cap S$.

- *Claim 5.* For all $i \in I$, $x_i$ is $\succeq_{i,z}$-maximal in the budget set.

  **Proof** Since the income of each consumers is positive, preferences are continuous and $0 \in X_i(z)$ for all $i$, we find by standards arguments that a strictly preferred consumption with respect to $x_i$ is out of the budget set.

## 5 When $L_+$ Has an Empty Interior

Now we address the case where $L$ is a Banach lattice whose positive cone has an empty interior. Nevertheless, we recall that we assume that $e$ is a quasi-interior point of $L_+$, that is, $L(e)$, the principal ideal generated by $e$, is norm dense in $L$. As well known in the literature, a properness condition becomes a key assumption in order to get a competitive equilibrium vector price. One of the most general properness definitions are those of [23] pp. 582-83.

For the consumption side:

Let $K$ be some order ideal of $L$. For all $i \in I$ and all $z \in L^{I+J}$, the preference relation $\succ_{i,z}$ is said to be $F$-proper relative to $K$ at $x \in X_i(z)$ if there exists a $\|\cdot\|$-open subset $V_x$ of $L$, a lattice $Z_x \subset K$ which verifies $Z_x + K_+ \subset Z_x$ and some subset $A_x$ of $L$, radial at $x$, such that $x \in \bar{V}_x \cap Z_x$ and $\emptyset \neq V_x \cap Z_x \cap A_x \subset conv\{x' \in X_i(z) : \ x' \succ_{i,z} x\}$

As for the production sector, [23] proposed the following condition:

Let $K$ be some order ideal of $L$. For all $z \in Z$, the technology $Y_j(z)$ is said to be $F$-proper relative to $y$ at $Y_j(z)$ if there exists a $\|\cdot\|$-open subset $V_y$ of $L$, a lattice $Z_y \subset K$ verifying $Z_y - K_+ \subset Z_y$ and some subset $A_y$ of $L$, radial at $y$, such that $y \in \bar{V}_y \cap Z_y$ and $\emptyset \neq V_y \cap Z_y \cap A_y \subset Y_j(z)$

These definitions look somewhat abstract since they encompass many properness conditions previously stated in the literature. We refer to [23] in order to observe some economic meaning of these conditions.

From the last definition, we can derive a particular uniform condition which is suitable for non-convex technologies:

**Uniform $e$-properness relative to $L(e)$**

We shall say that the technology $Y_j$ is uniformly proper relative to $L(e)$ if there exists a real number $\delta_j > 0$, which generates the open cone at zero $\Gamma_j = \{\alpha\xi : \alpha > 0, \xi \in \{-e\} + B(0, \delta_j)\}$ such that for all $z \in Z$ and all $\bar{y}_j \in Y_j(z)$, $(\{\bar{y}_j\} + \Gamma_j) \cap L(e) \subset Y_j(z)$. The element $e$ is the proper vector.

In words, we are saying that if $y_j$ is producible and we add to it the quantity $e$ of inputs, then it is still producible if we add a vector which is small enough and the resultant vector is order bounded by some multiple of $e$. Thus, marginal rates of substitution with respect to $e$ are bounded away from zero. In other words, the trace on $L(e)$ of the extended production set $Y_j(z) + \Gamma_j$ is included in $Y_j(z)$. Notice that since $B(0, \delta_j)$ does not depend on $y_j$ our condition is *uniform* instead of *pointwise*. By comparing with the $F$-properness condition of [23] relative to $y_j \in Y_j(z)$, note that in our case $K = Z_{y_j} = L(e)$, $V_{y_j} = \{y_j\} + \Gamma_j$ and $A_{y_j} = L$. We notice that if $e \in \text{int}L_+$ then uniform properness technology relative to $L(e)$ follows directly from the free-disposal Assumption since there would be a real number $\delta_j > 0$, such that $-e + B(0, \delta_j) \subset -L_+$ and $L(e) = L$.

## 5.1 The main existence theorem

**Theorem 11** *Let $\mathcal{E}$ be a Banach lattice economy. There exists a vector $(\bar{z}, \bar{\pi})$ in $Z \times \hat{\mathcal{N}}_{Y_j(z)}(y_j) \cap S$ which is a marginal pricing equilibrium if $\mathcal{E}$ satisfies Assumptions* (C), (P), (B), (SA), (TC) *and each technology is $e$-uniformly proper with respect to $L(e)$.*

Theorem 11 extends Theorem 7 since we do not impose any interiority assumption. Apart from the ordered preferences we have assumed, our result is at the same level of generality than the other existence results for competitive equilibrium whereas we encompass two major market imperfections, increasing returns and externalities.

## 5.2 Proof

Let $L(e)$ be the principal ideal generated by $e$. Hence the order interval $[-e, e]$ is radial in $L(e)$ and then the gauge of the set $[-e, e]$ induces a norm topology on $L(e)$. We call it the $\|\cdot\|_e$-topology. Actually, $[-e, e]$ is the closed unit ball on $L(e)$ while $B^e(0, 1) = \{x \in L(e) : \|x\|_e < 1\}$ is the open unit ball on $L(e)$. Let $L(e)^*$ denote the $\|\cdot\|_e$-dual of $L(e)$ and let $\|\cdot\|_e^*$ denote the dual norm on $L(e)^*$.

Let $L(e)_+ = L_+ \cap L(e)$. Clearly $L(e)_+$ is $\|\cdot\|_e$-closed in $L(e)$ and has a non-empty $\|\cdot\|_e$-interior which contains $e$. Obviously, $\|e\|_e = 1$. The $\|\cdot\|_e$-topology is finer than the topology of $L(e)$ as a subspace of $(L, \|\cdot\|)$ with the norm denoted $\|\cdot\|_{L(e)}$. Let $\tau_e$ be the restriction to $L(e)$ of the topology $\tau$. Clearly $\tau_e$ is included in the $\|\cdot\|_{L(e)}$-topology, which itself is included in the $\|\cdot\|_e$-topology. Furthermore, it is straightforward to check that $\tau_e$ is a Hausdorff and locally convex-solid topology such that all order intervals in $L(e)$ are $\tau_e$-compact.

Let $X_i^e : L(e)^{I+J} L(e)_+$ be the restricted consumption correspondence such that for all $z \in L(e)^{I+J}$, $X_i^e(z) = X_i(z) \cap L(e)_+$. $Y_j^e : L(e)^{I+J} L(e)$ is the restricted production correspondence such that for all $z \in L(e)^{I+J}$, $Y_j^e(z) = Y_j(z) \cap L(e)$. We also restrict properly the preference relation by $\succeq_{i,z}^e$ for all $z \in L(e)^{I+J}$. We remark that $\partial Y_j^e(z) \subset \partial Y_j(z) \cap L(e)$, whence $Z^e \subset Z$ and $A^e(\omega') = A(\omega') \cap Z^e \subset A(\omega')$. The revenue functions $(r_i)$ are the same. $p \in L(e)^*$ is a $\|\cdot\|_e$-continuous linear functional on $L(e)$. Finally, we have $PE^e = \{(p,z) \in L(e)^* \times Z^e : p \in \cap_{j \in J} \hat{\mathcal{N}}_{Y_j^e(z)}(y_j)\}$ and for all $z \in L(e)^{I+J}$, $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$ and $\hat{\mathcal{N}}_{Y_j^e(z)}(y_j)$ are the induced cones in $L(e)$.

The economy $\mathcal{E}^e$ is fully described by $((X_i^e, \succeq_{i,z}^e, r_i)_{i \in I}, (Y_j^e)_{j \in J}, \omega)$.

We show that Theorem 7 applies for $\mathcal{E}^e$. Clearly, for all $z \in L(e)^{I+J}$ the $\|\cdot\|_e$-interior of $X_i^e(z)$ is non-empty because of Assumption C(1) on solidity condition and the fact that $\delta e \in X_i^e(z)$ for all $\delta > 0$. $X_i^e$ satisfies the remaining of Assumption C(1) and Assumption C(2) and since the $\|\cdot\|_e$-topology is stronger than the $\|\cdot\|_{L(e)}$-topology, $G_i^e(z)$ is $\|\cdot\|_e \times \tau_e \times \prod_{L(e)^{I+J}} \tau_e$-closed in $L(e)^{2+I+J}$. As for non satiation, note that it remains true in $\mathcal{E}^e$ since in Assumption C(3), the improving consumption $(x_i')$ is chosen in $L(e)^I$. The next lemma whose proof is given in Appendix shows that Assumption SA holds true in the economy $\mathcal{E}^e$

**Lemma 12** *The economy $\mathcal{E}^e$ satisfies Assumption* (SA).

With respect to the production sector, it is not difficult to observe that Assumption P is fully satisfied on every $Y_j^e$. On the other hand, Assumption B is automatically satisfied. Finally one easily checks that Assumption R holds on $\mathcal{E}^e$.

Consequently, all conditions of Theorem 7 are satisfied. Thus, there is an equilibrium $((x_i)_{i \in I}, (y_j)_{j \in J}, p) \in Z^e \times \hat{\mathcal{N}}_{Y_j^e(z)}(y_j) \cap S^{e^*}$. For the next two claims we remark that $\hat{\mathcal{T}}_{Y_j(z)}(y_j) \cap L(e) \subset \hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$

**Proposition 13** *There exists a price functional $\pi \in L^*$ such that $\pi_{|L(e)} = p$*

**Proof** Let $\delta > 0$ be a real number such that $\delta < \delta_j$ for all $j \in J$, where $\delta_j$ are the parameters coming from uniform $e$-properness assumption relative to $L(e)$ on each production correspondence $Y_j$. Since $L$ is a locally convex topological vector space, we can apply a suitable version of the Hahn-Banach theorem (for instance, see [31], Theorem 4.2, p. 49) to show the existence of a continuous linear functional $\pi$ which extends $p$ to the whole space $L$. So, we have to show first that $p$ is $\|\cdot\|$-continuous on $L(e)$. We stress that accordingly to the above remark $p$ is $\|\cdot\|_e$-continuous. We now prove that the functional $p$ is bounded on $B(0,\delta)\cap L(e)$. Since $B(0,\delta)$ is circled (then symmetric) it suffices to prove it for any vector in $B(0,\delta)\cap L(e)_+$. Let $\xi \in B(0,\delta)\cap L(e)_+$. There exists $n_0 \in$ such that $\bar{\xi} = \frac{1}{n_0}\xi \leq e$. $\bar{\xi} \in L(e)\cap B(0,\frac{\delta}{n_0})$. We claim $-\frac{1}{n_0}e+\bar{\xi}$ belongs to $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$. Indeed, recall that by Assumption TC, $0 \in \hat{\mathcal{T}}_{Y_j(z)}(y_j)$ and then in $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$. Thus, for $\rho > 0$, there exists $\eta > 0$ such that for all $r > 0$ there exist relative neighborhoods $V^e \in L(e)^{I+J}$ (for $V \in_{\prod_{L^{I+J}}\tau} (z)$) and $U^e \in L(e)$ (for $U \in_\tau (\bar{y}_j)$) together with the real number $\varepsilon > 0$ such that for all $z' \in (\{z\}+\rho B^e(0,1)^{I+J})\cap V^e$, for all $\bar{y}'_j \in (\{y_j\}+\rho B^e(0,1))\cap U^e)\cap Y_j(z')$ and for all $t \in ]0,\varepsilon[$, $\bar{y}'_j + t(\eta(y_j-\bar{y}'_j)-re) \in Y_j(z')$. We note that $-t\frac{1}{n_0}e+t\bar{\xi} \in \Gamma_j$ for $t>0$ and then, by the $e$-uniform properness relative to $L(e)$, it follows that $\bar{y}'_j+t(\eta(y_j-\bar{y}'_j)-re)-t\frac{1}{n_0}e+t\bar{\xi} = \bar{y}'_j+t(-\frac{e}{n_0}+\bar{\xi}+\eta(y_j-\bar{y}'_j)-re) \in Y(z')\cap L(e)$. Thus we deduce that $-\frac{1}{n_0}e+\bar{\xi}$ belongs to $\hat{\mathcal{T}}^\rho_{Y_j^e(z)}(y_j)$ and since this is true for all $\rho > 0$, we get that $-\frac{1}{n_0}e+\bar{\xi}$ belongs to $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$.

Since $p \in \hat{\mathcal{N}}_{Y_j^e(z)}(y_j)$ and $p(e) = 1$, it follows that $p(\bar{\xi}) \leq \frac{1}{n_0}$. Hence, $p(\xi) = n_0 p(\bar{\xi}) \leq 1$. We conclude that $p$ is $\|\cdot\|$-continuous on $L(e)$ and thus there exists $\pi \in L^*$ which extends $p$.

Because of the free-disposal assumption and the fact that $\pi(e) = p(e) = 1$, $\pi > 0$.

**Proposition 14** $\pi \in \cap_{j\in J}\hat{\mathcal{N}}_{Y_j(z)}(y_j)$.

**Proof** Let $\nu \in \hat{\mathcal{T}}_{Y_j(z)}(y_j) \subset \hat{\mathcal{T}}^\rho_{Y_j(z)}(y_j)$ for any $j$ and $\rho > 0$. If $\nu \in L(e)$ then $\pi(\nu) = p(\nu) \leq 0$ and we are done. Let us suppose that $\nu \notin L(e)$. From the definiton of $\hat{\mathcal{T}}^\rho_{Y_j(z)}(y_j)$ we know that there exists $\eta > 0$ such that for $r > 0$ arbitrary there exist neighborhoods $V \in_{\prod_{L^{I+J}}\tau} (z)$, $U \in_\tau (y_j)$ and a real number $\varepsilon > 0$ such that for all $z' \in (\{z\}+\rho B^e(0,1)^{I+J})\cap V \cap L(e)$, for all $\bar{y}'_j \in B^e(y_j,\rho)\cap U \cap L(e)\cap Y_j(z')$ and for all $t \in ]0,\varepsilon[$ there exists $\xi \in r[-e,\ e]$ such that $\bar{y}'_j + t(\nu+\eta(y_j-\bar{y}'_j)+\xi) \in Y(z')$.

Let us choose $\beta > 0$. Let $\delta_j > 0$ be the parameter coming from uniform $e$-properness relative to $L(e)$. Since $\pi$ is $\|\cdot\|$-continuous, there exists $\delta' > 0$ such that $B(0,\delta') \subset \{x \in L : |\pi(x)| < \beta\}$. Let $\delta > 0$ such that $\delta < \beta\delta_j$ and $\delta < \delta'$. Since $L(e)$ is norm-dense in $L$ there exists $u \in B(0,\delta)$ such that $\nu + u \in L(e)$. Hence, $\bar{y}'_j+t(\nu+\eta(y_j-\bar{y}'_j)+\xi)-t\beta e+tu = \bar{y}'_j+t(\nu-\beta e+u+\eta(y_j-\bar{y}'_j)+\xi) \in L(e)$. By uniform $e$-properness, $\bar{y}'_j + t(\nu-\beta e+u+\eta(y_j-\bar{y}'_j)+\xi) \in Y_j(z')\cap L(e)$. This in turn implies $\nu-\beta e+u \in \hat{\mathcal{T}}^\rho_{Y_j^e(z)}(y_j)$ and since this is true for all $\rho > 0$ we have $\nu-\beta e+u \in \hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$. Hence $\pi(\nu-\beta e+u) = p(\nu-\beta e+u) \leq 0$, and thus $\pi(\nu) \leq 2\beta$. This inequality is true for all $\beta > 0$ and $j \in J$ and thus Proposition14 is proved.

The last part of the proof consists in proving that every consumer is in equilibrium according to her/his budget constraint and preferences. Let $i \in I$ and $x'_i \in X_i(z)$ such that $x'_i \succ_{i,z} x_i$ and $\pi(x'_i) < \pi(x_i)$. Let us note that

$(x_i, x'_i, z) \notin G_i$. By Lemma 3 in [8], $L(e)_+$ is norm-dense in $L_+$ and thus one can choose a parameter $\varepsilon > 0$ small enough such that there exists $u \in B(0, \varepsilon)$, $x'_i + u \in L(e)_+$, $(x_i, x'_i + u, z) \notin G_i$ by Assumption C(4) and $\pi(x'_i + u) < \pi(x_i)$. Since there exists $n_0 \in N$ for which $x'_i + u \leq n_0 e$ we deduce by solidity of $X_i(z)$ (Assumption C(1)) and Assumption C(2) that $x'_i + u \in X_i(z)$. Consequently, $\pi(x'_i + u) = p(x'_i + u) < \pi(x_i) = p(x_i)$, which contradicts the fact that $x_i$ is an equilibrium consumption vector for $i$ in $\mathcal{E}^e$

# 6 Conclusions

In this paper, we provide a new definition for a tangent cone and its polar cone, the normal cone, in Banach lattices. Basically, the key idea remains the same, namely, defining a first order necessary condition for profit maximization but taking into account the order relation through the use of order intervals. The class of Banach lattices allows to consider most of the commodity spaces already considered in the literature beyond the Euclidean finite dimension spaces and the space of essentially bounded, real valued and measurable functions. Our definition fits with the previous concepts used in the literature; when the production sets are smooth, when they are defined by a convex valued correspondence, when the interior of the positive cone is nonempty, when the commodity space is finite dimensional, when there is no externality. But the key property of this normal cone is the fact that it is compatible with the existence of an equilibrium under assumptions at the same level of generality as for a competitive equilibrium. Note that we only assume uniform properness on technologies and not on consumers which differs, substantially, from what has been done in competitive production economies.

The next step for further research is the computation of this normal cone when the production sets are defined: by several smooth inequalities representing different transformation functions; in particular spaces like $L^2$ for model with uncertainty with transformation functions representing the mean expectation for the outcome in the different states of the world; when the transformation function is defined recursively for intertemporal models with infinite horizon. Another line of research is the computation of the normal cone of a production set, which is obtained from the aggregation of two productions sets having a reciprocal external effect. Indeed, a standard way of fighting against the lack of optimality coming from externalities is to merge the producers so that the external effects are internalized. But, the global production set is likely non convex even if we start with two producers defined by a convex-valued correspondence. That is why it is important to compute the new normal cone to define at least a necessary condition for optimality.

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Appendix

# Proof of Proposition 2

We first recall the definition of the Clarke's tangent and normal cones to the set $Y_j(z)$ at the point $\bar{y}_j$

$$T_{Y_j(z)}(\bar{y}_j) := \left\{ \nu \in L : \begin{array}{l} \forall r > 0 \; \exists \; \varepsilon > 0 : \forall y'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z), \forall t \in ]0, \varepsilon[ \\ {[y'_j + tB(\nu, r)]} \cap Y(z) \neq \emptyset \end{array} \right\}$$

$N_{Y_j(z)}(\bar{y}_j) = [T_{Y_j(z)}(\bar{y}_j)]^o = \{p \in L^* : p(\nu) \leq 0 \; \forall \nu \in T_{Y_j(z)}(\bar{y}_j)\}$

$\hat{T}_{Y_j(z)}(\bar{y}_j)$ is non-empty since the free-disposal condition implies that $-L_+ \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$. Now we show that $\hat{T}_{Y_j(z)}(\bar{y}_j)$ is a cone. Let $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$ and $\tau > 0$. Let $r > 0$ and $\varepsilon$ be the parameter associated by the definition of $\hat{T}_{Y_j(z)}(\bar{y}_j)$ to $\frac{r}{\tau}$. Hence, for all $z' \in B(z, \varepsilon)$, for all $\bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z')$ and for all $t \in ]0, \varepsilon[$ there exists $\xi \in B(0, \frac{r}{\tau})$ such that $\bar{y}'_j + t(\nu + \xi) \in Y_j(z')$. Let $\varepsilon'$ strictly smaller than $\varepsilon$ and $\frac{\varepsilon}{\tau}$. Hence, for every $z' \in B(z, \varepsilon')$ for every $\bar{y}'_j \in B(\bar{y}_j, \varepsilon')$ and for every $t \in ]0, \varepsilon'[$, since $t\tau < \varepsilon$, there exists $\xi \in B(0, \frac{r}{\tau})$ such that $\bar{y}'_j + \tau t(\nu + \xi) = \bar{y}'_j + t(\tau\nu + \tau\xi) \in Y_j(z')$. As $\tau\xi \in B(0, r)$, we have proved that $\tau\nu \in \hat{T}_{Y_j(z)}(y_j)$ by associating to $r$ the parameter $\varepsilon'$ and thus $\hat{T}_{Y_j(z)}(\bar{y}_j)$ is a cone.

We now show that $\hat{T}_{Y_j(z)}(\bar{y}_j) + \hat{T}_{Y_j(z)}(\bar{y}_j) \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$. Let $\nu$ and $\nu'$ be two vectors in $\hat{T}_{Y_j(z)}(\bar{y}_j)$. For $r > 0$ there exist two non negative real numbers $\varepsilon$ and $\varepsilon'$ associated by the definition of $\hat{T}_{Y_j(z)}(\bar{y}_j)$ to $\frac{r}{2}$. Let $\varepsilon_1 > 0$ smaller than $\varepsilon$ and $\frac{\varepsilon'}{1+\|\nu\|+\frac{r}{2}}$. Hence, for all $z' \in B(z, \varepsilon_1)$, for all $\bar{y}'_j \in B(\bar{y}_j, \varepsilon_1) \cap Y_j(z')$ and for all $t \in ]0, \varepsilon_1[$, there exists $\xi \in B(0, \frac{r}{2})$ such that $\bar{y}''_j = \bar{y}'_j + t(\nu + \xi) \in Y_j(z')$. We remark that $\|\bar{y}''_j - y_j\| \leq \|\bar{y}'_j - \bar{y}_j\| + t(\|\nu\| + \|\xi\|) < \varepsilon_1 + \varepsilon_1(\|\nu\| + \frac{r}{2}) \leq \varepsilon'$. Consequently, since $\varepsilon_1 < \varepsilon'$, there exists $\xi' \in B(0, \frac{r}{2})$ such that $\bar{y}''_j + t(\nu' + \xi') \in Y_j(z')$. Hence, $\bar{y}'_j + t(\nu + \nu' + \xi + \xi') \in Y_j(z')$ and $\xi + \xi' \in B(0, r)$. So $\varepsilon_1$ associated to $r$ satisfies the defintion of $\hat{T}_{Y_j(z)}(\bar{y}_j)$ and we have shown that $\hat{T}_{Y_j(z)}(\bar{y}_j)$ is closed under addition.

The proof of the second part is trivial given the definition of the Clarke's normal cone.

## Proof of Proposition 3

**1.** $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ is non-empty by Assumption TC. We now prove that $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ is a cone. Let $\rho > 0$, $\tau > 0$ and $\nu \in \cap_{\rho > 0} \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j) \subset \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$. Then there exists $\eta > 0$ such that for all $r > 0$ there exist $V \in_{\prod_{L^{I+J}} \tau} (z)$, $U \in_{\tau} (\bar{y}_j)$ and $\varepsilon > 0$ such that for all $z' \in B(z, \rho) \cap V$, for all $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U \cap Y_j(z')$ and for all $t \in ]0, \varepsilon[$ it follows that $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) \in Y_j(z')$ for some $\xi \in r[-e, e]$. Let $\eta' = \tau\eta$. Let $r > 0$. Let $V, U$ and $\varepsilon$ the parameter and the open neighborhoods associated for $\nu$ to $\frac{r}{\tau}$. Let $\varepsilon' = \frac{\varepsilon}{\tau}$ and note that $\tau t \in ]0, \varepsilon[$ is equivalent to $t \in ]0, \varepsilon'[$. Hence, for all $z' \in B(z, \rho) \cap V$, for all $\bar{y}'_j \in B(\bar{y}_j + \rho) \cap U \cap Y_j(z')$ and for all $\tau t \in ]0, \varepsilon[$, there exists $\xi \in \frac{r}{\tau}[-e, e]$ such that $\bar{y}'_j + \tau t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) = \bar{y}'_j + t(\tau\nu + \eta'(\bar{y}_j - \bar{y}'_j) + \tau\xi) \in Y_j(z')$ with $\tau\xi \in r[-e, e]$. Hence, $\tau\nu \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$. Since this holds for all $\rho > 0$, $\tau\nu \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$

The normal cone $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$ is weak*-closed since it is the intersection of weak*-closed half spaces $\{\pi \in L^* : \pi(\nu) \leq 0\}$ over the $\nu$ in $\hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$.

**2.** From the definition of $\hat{\mathcal{T}}$, it is enough to prove that $\hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j) \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$ for all $\rho > 0$. Let $\rho > 0$ and $\nu \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$. Consequently, there exists $\eta > 0$ such that for all $r > 0$ there exist $V \in_{\prod_{L^{I+J}} \tau} (z)$, $U \in_{\tau} (\bar{y}_j)$ and $\varepsilon > 0$ associated to $\frac{r}{2}$. Let us fix $\varepsilon' > 0$ strictly smaller than $\varepsilon, \frac{r}{2\eta}, \rho$ and such that $B(z, \varepsilon') \subset B(z, \rho) \cap V$ and $B(\bar{y}_j, \varepsilon') \subset B(\bar{y}_j, \rho) \cap U$. Thus, for all $z' \in B(z, \varepsilon')$, for all $\bar{y}'_j \in B(\bar{y}_j, \varepsilon') \cap Y_j(z')$ and for all $t \in ]0, \varepsilon'[$, we get the existence of a vector $\xi \in \frac{r}{2}[-e, e]$ such that $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) \in Y_j(z')$. Note that $\|\eta(\bar{y}_j - \bar{y}'_j)\| \leq \frac{r}{2}$ and thus $\|\xi' = \xi + \eta(\bar{y}_j - \bar{y}'_j)\| \leq r$ and $\bar{y}'_j + t(\nu + \xi') \in Y_j(z')$. Consequently, $\hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j) \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$.

By polarity we obtain that $\hat{N}_{Y_j(z)}(\bar{y}_j) \subset \hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$.

**3.** $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) \subset \mathcal{T}_{Y_j(z)}(\bar{y}_j)$ since order intervals are norm-bounded thanks to the fact that the norm is a lattice norm.

**4.** Since $e \in \int L_+$, as already mentioned in Section 3, without any loss of generality, we choose the norm as the lattice norm associated to $e$. So $\bar{B}(0,1) = [-e, e]$ and we can replace it in the definition of $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$, $\xi \in r[-e, e]$ by $\xi \in B(0, r)$.

We show that $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) + \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) \subset \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ in order to prove the convexity. Let $\nu$ and $\nu'$ be in $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$. Hence they belong to $\hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$ for every $\rho > 0$. Let us consider the set $\hat{\mathcal{T}}^{\rho+1}_{Y_j(z)}(\bar{y}_j)$. There exist $\eta$ and $\eta'$ such that for every

$r>0$ there exist neighborhoods $\varepsilon, \varepsilon', U, U', V$ and $V'$ associated to the real number $\frac{r}{3}$, such that for all $z' \in B(z,\rho+1)\cap V$, for all $\bar{y}'_j \in B(\bar{y}_j,\rho+1)\cap U \cap Y_j(z')$ and for all $t \in ]0,\varepsilon[$, it follows that $\left[\{\bar{y}'_j\} + tB(\nu+\eta(\bar{y}_j-\bar{y}'_j),\frac{r}{3})\right]\cap Y_j(z') \neq \emptyset$ and for all $z' \in B(z,\rho+1)\cap V'$, for all $\bar{y}'_j \in B(\bar{y}_j,\rho+1)\cap U' \cap Y_j(z')$ and all $t \in ]0,\varepsilon'[$, $[\{\bar{y}'_j\} + tB(\nu+\eta'(\bar{y}_j-\bar{y}'_j),\frac{r}{3})]\cap Y_j(z') \neq \emptyset$. There exist $\alpha \in ]0,1[$, $V''$ and $U''$ such that $V''+B(0,\alpha)\subset V\cap V'$ and $U''+B(0,\alpha)\subset U\cap U'$. Let $\varepsilon''>0$ strictly smaller than $\varepsilon, \varepsilon', \frac{r}{\eta'(3\|\nu\|+3\rho\eta+r)}$ and $\frac{3\alpha}{3\|\nu\|+3\rho\eta+r}$. Hence, for every $z'$ in $B(z,\rho)\cap V'' \subset B(z,\rho+1)\cap V$, for every $\bar{y}'_j$ in $B(y,\rho)\cap U''\cap Y_j(z') \subset B(\bar{y}_j,\rho+1)\cap U\cap Y_j(z')$ and for every $t$ in $]0,\varepsilon''[ \subset ]0,\varepsilon[$, there exists $\xi$ in $B(0,\frac{r}{3})$ in such a way that the vector $\zeta_j = \bar{y}'_j + t(\nu+\eta(\bar{y}_j-\bar{y}'_j)+\xi)$ belongs to $Y_j(z')$. From the definition of $\varepsilon''$, one easily checks that $\|\zeta_j - \bar{y}'_j\| < \alpha < 1$ and $\eta'\|\zeta_j - \bar{y}'_j\| \leq \frac{r}{3}$. So, $\zeta_j \in B(\bar{y}_j,\rho+1)$ and $\|t(\nu+\eta(\bar{y}_j-\bar{y}'_j)+\xi\| < \alpha$. Consequently, since $\bar{y}'_j \in U''$ and $z' \in B(z,\rho+1)\cap V'$, there exists $\xi' \in L$ such that $\|\xi'\| < \frac{r}{3}$ and $\zeta'_j = \zeta_j + t(\nu'+\eta'(\bar{y}_j-\zeta_j)+\zeta'_j) \in Y_j(z')$. We note that the vector $\zeta'_j$ equals $\bar{y}'_j + t(\nu+\nu'+(\eta+\eta')(\bar{y}_j-\bar{y}'_j)+\xi+\xi'+\eta'(\bar{y}'_j-\zeta_j))$ and $\|\eta'(\bar{y}'_j-\zeta_j)\|$ is strictly smaller than $\frac{r}{3}$. Hence, $\left[\{\bar{y}'_j\} + tB(\nu+\nu'+(\eta+\eta')(\bar{y}_j-\bar{y}'_j),r)\right]\cap Y_j(z') \neq \emptyset$ and thus $\nu+\nu' \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$. Since this is true for all $\rho>0$, we conclude that $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ is stable under sumation. Since $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ is a cone, we get it is convex.

**5.** We first show that when $L$ is finite dimensional $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) = \hat{T}_{Y_j(z)}(\bar{y}_j)$. Let $r>0$ and $\varepsilon$ associated with $\frac{r}{2}$ such that for all $z' \in B(z,\varepsilon)$, for all $\bar{y}'_j \in B(\bar{y}_j,\varepsilon)\cap Y_j(z')$ and for all $t \in ]0,\varepsilon[$ there exists $\xi \in B(0,\frac{r}{2})$ such that $\bar{y}'_j + t(\nu+\xi) \in Y_j(z')$. Let $\rho>0$ and $\eta>0$. Since in this case all topologies are equivalents and the interior of the positive cone is nonempty, we can choose the closed unit ball equal to the order interval $[-e,e]$. We now choose $\varepsilon'>0$ smaller than $\varepsilon$, $\frac{r}{2\eta}$ and $\rho$. Let $V = B(z,\varepsilon')$ and $U = B(\bar{y}_j,\varepsilon')$. Then, for all $\bar{y}'_j \in U$ one has that $\|\xi - \eta(\bar{y}_j-\bar{y}'_j)\| < r$ for all $\xi \in B(\nu,\frac{r}{2})$. This implies that for all $z' \in V\cap B(z,\rho)$, for all $\bar{y}'_j \in U\cap B(\bar{y}_j,\rho)\cap Y_j(z')$ and all $t \in ]0,\varepsilon'[$, there exists $\xi \in B(0,\frac{r}{2})$ such that $\bar{y}'_j + t(\nu+\xi) \in Y_j(z')$. But $\bar{y}'_j + t(\nu+\xi) = \bar{y}'_j + t(\nu+\eta(\bar{y}_j-\bar{y}'_j)-\eta(\bar{y}_j-\bar{y}'_j)+\xi)$. Since $\xi-\eta(\bar{y}_j-\bar{y}'_j)\in B(0,r)$, $\nu \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$, so $\nu \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ since this is true for all $\rho>0$.

Let us now consider the following sequential characterization of $\hat{T}_{Y_j(z)}(\bar{y}_j)$ when $L$ is a finite dimensional vector space:

$$\hat{T}_{Y_j(z)}(\bar{y}_j) := \left\{ \nu \in L : \begin{array}{c} \forall z^n \to z, \forall y^n_j \to \bar{y}_j \text{ such that } y^n_j \in \partial Y_j(z^n) \\ \text{and } \forall t^n \downarrow 0 \ \exists \nu^n \to \nu \text{ such that } y^n_j + t^n\nu^n \in Y_j(z^n) \end{array} \right\}$$

Let us recall the marginal pricing rule of [16] when the commodity space is of finite dimension:

$$MP(\bar{y}_j,z) = conv \left\{ p \in S : \begin{array}{c} \exists\ z^n \subset^{L(I+J)}, z^n \to z, \\ \exists\ \bar{y}^n_j \subset^{L}, \bar{y}^n_j \to \bar{y}_j,\ \bar{y}^n_j \in \partial Y_j(z^n) \\ \exists\ p^n \subset^{L}, p^n \to p,\ p^n \in N_{Y_j(z^n)}(\bar{y}^n_j) \cap S \end{array} \right\}$$

To prove that $\hat{N}_{Y_j(z)}(\bar{y}_j) \cap S = MP(\bar{y}_j,z)$, we first prove the following lemma:

**Lemma 15** *(i) Let* $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$ *and let* $r > 0$ *and* $\varepsilon > 0$ *as given by the definition of* $\hat{T}_{Y_j(z)}(\bar{y}_j)$. *For all* $z' \in B(z,\varepsilon)$ *and all* $\bar{y}'_j \in B(\bar{y}_j,\varepsilon) \cap Y_j(z')$, $\nu - re \in \hat{T}_{Y_j(z')}(\bar{y}'_j)$.

*(ii) Let* $\nu \in MP(\bar{y}_j,z)^\circ$, *for all* $\delta > 0$, *there exists* $\varepsilon > 0$ *such that for all* $z' \in B(z,\varepsilon)$ *and all* $\bar{y}'_j \in B(\bar{y}_j,\varepsilon) \cap Y_j(z')$, $\nu - \delta e \in \int T_{Y_j(z')}(\bar{y}'_j)$.

**Proof.** *(i)* Let $\nu \in \hat{T}_{Yj(z)}(\bar{y}_j)$. Let us choose $\varepsilon' < \varepsilon$ such that $B(z',\varepsilon') \subset B(z,\varepsilon)$ and $B(\bar{y}'_j,\varepsilon') \subset B(\bar{y}_j,\varepsilon)$. Then, for all $z'' \in B(z',\varepsilon')$ for all $\bar{y}''_j \in B(\bar{y}'_j,\varepsilon') \cap Y_j(z'')$ and all $t \in (0,\varepsilon')$ there exists $\xi \in B(0,r)$ such that $\bar{y}''_j + t(\nu+\xi) \in Y_j(z'')$. By the free disposal condition, $\bar{y}''_j + t(\nu - re) \in Y(z'')$. From the definition of $\hat{T}_{Y_j(\cdot)}$, we have $\nu - re \in \hat{T}_{Y_j(z')}(\bar{y}'_j)$.

*(ii)* Let $\nu \in MP(\bar{y}_j,z)^\circ$ and $\delta > 0$. By contraposition, if for all $\varepsilon > 0$, there exist vectors $z' \in B(z,\varepsilon)$ and $\bar{y}'_j \in B(\bar{y}_j,\varepsilon) \cap Y_j(z')$ such that $\nu - \delta e \notin \int T_{Y_j(z')}(\bar{y}'_j)$, we can build a sequence $(z^n, \bar{y}^n_j, p^n)$ converging to $(z, \bar{y}_j, p)$ such that for each $n$, $\bar{y}^n_j \in \partial Y_j(z^n)$, $p^n \in N_{Y_j(z^n)}(\bar{y}^n_j) \cap S$ and $\langle p^n, \nu - \delta e\rangle \geq 0$. So, at the limit, from the definition of $MP(\bar{y}_j,z)$, we get $p \in MP(\bar{y}_j,z)$ and $\langle p, \nu - \delta e\rangle \geq 0$, which implies $\langle p, \nu\rangle \geq \delta > 0$, which contradicts $\nu \in MP(\bar{y}_j,z)^\circ$. ∎

We now proceed with the proof. Let $\pi \in MP(\bar{y}_j,z)$, then by Carathéodory's Theorem $\pi = \sum_{k\in K} \lambda_k p_k$ such that $\lambda_k \geq 0$, $\sum_{k\in K} \lambda_k = 1$, and $(p_k) \in S$ for all $k$. Hence, for all $\delta > 0$, $\langle \pi, \nu - \delta e\rangle = \sum_{k\in L} \lambda_k \langle p_k, \nu - \delta e\rangle$. By definition, every $p_k$ is the limit of a sequence $p^n_k \in N_{Y(z^n)}(\bar{y}^n_j) \cap S \subset \hat{N}_{Y(z^n)}(\bar{y}^n_j) \cap S$ from Proposition 2. Let $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$. For all $n \geq n_0$ $\sum_{k\in K} \lambda_k \langle p^n_k, \nu - \delta e\rangle \leq 0$ by the above lemma. Taking limits, one has $\langle \pi, \nu - \delta e\rangle = \sum_{k\in K} \lambda_l \langle p_k, \nu - \delta e\rangle \leq 0$. Hence, $\langle \pi, \nu\rangle \leq \delta$. Since this is true for all $r > 0$, we conclude that $\langle \pi, \nu\rangle \leq 0$ and then $\pi \in \hat{N}_{Y_j(z)}(\bar{y}_j) \cap S$.

To prove the converse inclusion, we use the duality between closed convex cones and we actually prove that $MP(\bar{y}_j,z)^\circ \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$. Let $\nu \in MP(\bar{y}_j,z)^\circ$. To prove $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$, it suffices to show that $\nu - \delta e \in \hat{T}_{Y_j(z)}(\bar{y}_j)$ for all $\delta > 0$. From the above lemma, for all $\delta > 0$, there exists $\varepsilon > 0$ such that for all $z' \in B(z,\varepsilon)$ and all $\bar{y}'_j \in B(\bar{y}_j,\varepsilon) \cap Y_j(z')$, $\nu - \delta e \in \int T_{Y_j(z')}(\bar{y}'_j)$. So, from the characterisation of the interior of the Clarke' tangent cone, there exists $\tau(\bar{y}'_j, z') > 0$ such that for all $t \in [0, \tau(\bar{y}'_j, z)]$, $\bar{y}'_j + t(\nu - \delta e) \in Y_j(z')$. Let $z^n \to z$ and $\bar{y}^n_j \to \bar{y}_j$ such that $\bar{y}^n_j \in \partial Y_j(z^n)$. For $n$ large enough, $z^n \in B(z,\varepsilon)$ and $\bar{y}^n_j \in B(\bar{y}_j,\varepsilon) \cap Y_j(z')$. So, we can build a sequence $t^n \downarrow 0$ such that $t^n < \tau(\bar{y}^n_j, z^n)$. Hence, $\bar{y}^n_j + t^n(\nu - \delta e) \in Y_j(z^n)$ for all $n$ large enough, which implies that $\nu - \delta e \in \hat{T}_{Y_j(z)}(\bar{y}_j)$.

**6.** Let $Y_j$ be a convex valued correspondence. Let

$$PM(\bar{y}_j, z) = \{\pi \in L^* : \pi(\bar{y}_j) \geq \pi(y'_j) \forall\ y'_j \in Y_j(z)\}$$

be the profit maximization behaviour. Let $\zeta_j - \bar{y}_j \in (Y_j(z) - \{\bar{y}_j\})$. Let $\rho > 0$, $\eta = 1$, $r > 0$, $0 < \delta < r$, $\varepsilon = 1$ and $U = L$. By Assumption P(3), there exists $V \in \prod_{L^M} \tau$ $(z)$ such that $\zeta_j - \delta e \in Y_j(z')$ for all $z' \in V$. Let $z' \in B(z,\rho) \cap V$ and $\zeta_j - \delta e \in Y_j(z')$. Then, for $\bar{y}'_j \in B(\bar{y}_j,\rho) \cap Y_j(z')$ and

$t \in ]0,1[$, $t(\zeta_j - \delta e) + (1-t)\bar{y}'_j \in Y_j(z')$ since $Y_j(z')$ is convex. But this means that $\bar{y}'_j + t(\zeta_j - \delta e - \bar{y}_j + (\bar{y}_j - \bar{y}'_j)) \in Y_j(z')$. Since $-\delta e \in r[-e,\ e]$ we have that $\zeta_j - \bar{y}_j \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$. Since this is true for all $\rho > 0$, $\zeta_j - \bar{y}_j \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ and thus $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \subset (Y_j(z) - \{\bar{y}_j\})^{\circ} = PM_j(\bar{y}_j, z)$. The converse is immediate since $PM(\bar{y}_j, z) = N_{Y_j(z)}(\bar{y}_j) \subset \hat{N}_{Y_j(z)}(\bar{y}_j) \subset \hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$.

# Proof of Lemma 4

By Assumption SB (2), for all $\zeta_j \in \partial Y_j(z)$, $T_{Y_j(z)}(\zeta_j) = \{\nu \in L | \nabla_1 f_j(\zeta_j, z)(\nu) \leq 0\}$ is the Clarke's tangent cone to $Y_j(z)$ at $\zeta_j$ ([14], Theorem 2.4.7, Corollary 2, p. 57).

Since $e$ is in the quasi-interior of $L_+$ and $\nabla_1 f_j(\zeta_j, z) \in L^*_+ \backslash \{0\}$, $\nabla_1 f_j(\zeta_j, z)(e)$ is strictly positive. Let $\nu \in L$ such that $\nabla_1 f_j(\zeta_j, z)(\nu) \leq 0$, let $\rho > 0$, $\eta = 1$ and $r > 0$. Let $\alpha > 0$ such that $\alpha < \frac{r\beta}{2(2\|\nu\|+4\rho+r)}$ where $\beta = \nabla_1 f_j(\zeta_j, z)(e)$. Since $\nabla_1$ is continuous, there exist neighborhoods $U \in_{\tau} (\zeta_j)$ and $V \in_{\prod_{L^{I+J}} \tau} (z)$ such that $\|\nabla_1 f_j(\zeta_j, z) - \nabla_1 f_j(\zeta'_j, z')\| < \alpha$ for all $(\zeta'_j, z') \in U \times V$.

Let $U' = \{\zeta'_j \in L \mid \nabla_1 f_j(\zeta_j, z)(\zeta'_j - \zeta_j) < \frac{r\beta}{4}\}$ be a weak neighborhood of $\zeta_j$. There exists another convex neighborhood $U''$ of $\zeta_j$ and $\delta > 0$ such that $U'' + B(0, \delta) \subset U' \cap U$. Let $\varepsilon > 0$ such that $\varepsilon < \frac{2\delta}{2(2\|\nu\|+2\rho+r)}$. From the Mean Value Theorem, for all $z' \in V \cap B(z, \rho)$, for all $\zeta'_j \in U'' \cap B(\zeta_j, \rho) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, there exists $\zeta''_j$ in the segment $[\zeta'_j, \zeta'_j + t(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e)]$ such that

$$f(\zeta'_j + t(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e), z') = f(\zeta'_j, z') + t\nabla_1 f_j(\zeta''_j, z')(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e)$$

From our choice of $\varepsilon$, it follows that $\zeta''_j \in U$, hence $\|\nabla_1 f_j(\zeta''_j, z') - \nabla_1 f_j(\zeta_j, z)\| \leq \alpha$. Since $\nabla_1 f_j(\zeta''_j, z')(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e) = (\nabla_1 f_j(\zeta''_j, z') - \nabla_1 f_j(\zeta_j, z))(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e) + \nabla_1 f_j(\zeta_j, z)(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e)$, we deduce from the previous definitions and inequalities that $\nabla_1 f_j(\zeta''_j, z')(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e) < 0$. Since $f(\zeta'_j, z') \leq 0$, we get $f(\zeta'_j + t(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e), z') \leq 0$, that is $\zeta'_j + t(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e) \in Y_j(z')$. Since $-\frac{r}{2}e \in [-re, re]$, we obtain that $\nu \in \hat{\mathcal{T}}^{\rho}{}_{Y_j(z)}(\zeta_j)$. Since this is true for all $\rho > 0$, we have that $\nu \in \hat{\mathcal{T}}_{Y_j(z)}(\zeta_j)$

# Proof of Lemma 8

- WSA$^F$

Suppose that $\mathcal{E}^F$ does no satisfy WSA$^F$, that is, for all $F$, $(p^F, z^F, \lambda^F)$ in $PE^F \times [0, \bar{\lambda}]$, $(y^F_j)_{j \in J} \in A(\omega + \lambda^F e, z^F)$ and $p^F(\sum_{j \in J} y^F_j + \omega + \lambda^F e) = 0$. By Lemma 10 there exists $\pi^F_j \in \hat{N}_{Y_j(z)}(\bar{y}_j) \cap S$ such that $\pi_{j|F} = p^F$ for all $j$. Since $y^F_j \in A(\omega + \bar{\lambda} e, z^F)$, Assumption (B) says that the net $(z^F, (\pi^F_j), (\pi^F_j(y^F_j)), \lambda^F)$ belongs to a weak-compact set such that the subnet $(z^{F(\gamma)}, (\pi^{F(\gamma)}), (\pi^{F(\gamma)}_j(y^{F(\gamma)}_j)), \lambda^{F(\gamma)})_{\gamma \in \Gamma}$ converges for the product topology to $(\bar{z}, (\bar{\pi}_j), lim(\pi^{F(\gamma)}_j(y^{F(\gamma)}_j)), \bar{\lambda})$

Since $z^{F(\gamma)} \in A(\omega + \lambda^{F(\gamma)}e)$, $\sum_{j\in J} y_j^{F(\gamma)} + \omega + \lambda^{F(\gamma)}e \geq \sum_{i\in I} x_i^{F(\gamma)}$ , and since $L_+$ is $\tau$-closed, it follows $\sum_{j\in J} \bar{y}_j + \omega + \lambda es \geq \sum_{i\in I} \bar{x}_i$. By Assumptions C(1) and P(1) we get $\bar{z} \in \prod_{i\in I} X_i(\bar{z}) \times \prod_{j\in J} Y_j(\bar{z})$ and by repeating the arguments of Claim 1 in Section 4.2, $\bar{\pi}_j = \bar{\pi} > 0$ for all $j \in J$. Consequently, $\sum_{j\in J} \bar{\pi}_j(\bar{y}_j) + \bar{\pi}_j(\omega) + \lambda \geq 0$. Given that $p^F(\sum_{j\in J} y_j^F + \omega + \lambda^F e)$ $= \pi_j^F(\sum_{j\in J} y_j^F + \omega + \lambda^F e) = 0$ for all $F \in \mathcal{F}$ and $j \in J$, we obtain that $lim\pi_j^{F(\gamma)}(y_j^{F(\gamma)}) + \bar{\pi}_j(\omega) + \lambda = 0$. By Proposition 5 $lim\pi_j^{F(\gamma)}(y_j^{F(\gamma)}) \geq \bar{\pi}(\bar{y}_j)$, so we deduce that $\bar{\pi}(\sum_{j\in J} \bar{y}_j + \omega + \lambda e) = \sum_{j\in J} lim\pi_j^{F(\gamma)}(y_j^{F(\gamma)}) + \bar{\pi}(\omega) + \lambda = 0$. Hence, $lim\pi_j^{F(\gamma)}(y_j^{F(\gamma)}) = \bar{\pi}(\bar{y}_j)$ for all $j$. From Proposition 5 b) $\bar{\pi} \in \hat{\mathcal{N}}_{Y_j(\bar{z})}(\bar{y}_j) \cap S$ and by Assumption (WSA) we get $\bar{\pi}(\sum_{j\in J} \bar{y}_j + \omega + \lambda e) > 0$, which contradicts the above equality.

- $(\text{LNS}^F)$

We show that there exists $\hat{F} \in \mathcal{F}$ such that for all $F \in \mathcal{F}$ with $\hat{F} \subset F$, the economy $\mathcal{E}^F$ satisfies $(\text{LNS}^F)$. We first prove that preferences are non-satiated. Suppose, on the contrary, that for all $F \in \mathcal{F}$ preferences are not satiated on the attainable allocations, i.e., for all $F \in \mathcal{F}$m there exists $z^F \in A^F(\omega)$ such that for some $i_0$, there does not exist $\xi_{i_0}^F \in X_{i_0}(z^F) \cap F$ such that $\xi_{i_0}^F \succ_{i_0,z^F}^F x_{i_0}^F$. Since $A^F(\omega) \subset A(\omega)$ for all $F$, there exists a subnet $(z^{F(\gamma)})_{\gamma\in\Gamma}$ converging weakly to $\bar{z} = ((\bar{x}_i)_{i\in I}, (\bar{y}_j)_{j\in J})$. From Assumptions C(1) and P(1) we deduce that $\bar{z} \in \prod_{i\in I} X_i(\bar{z}) \times \prod_{j\in J} Y_j(\bar{z})$ and from Assumption C(3) there exists a vector $(\xi_i)_{i\in I} \in \prod_{i\in I} X_i(\bar{z})$ such that $\xi_i \succ_{i,\bar{z}} \bar{x}_i$ for all $i$. Since the net $(z^{F(\gamma)})_{\gamma\in\Gamma}$ converges weakly to $\bar{z}$ and because of strong lower hemi-continuity of $X_i$ (Assumption C(2)), for $\delta > 0$, there exists $\gamma_0 \in \Gamma$ such that $\xi_i + \delta e$ belongs to $X_i(z^{F(\gamma)})$ for $\gamma > \gamma_0$. Further, there exists $F \in \mathcal{F}$ such that $\xi_i + \delta e \in F$ and $\gamma_1 \in \Gamma$ such that $F \subset F(\gamma)$ for all $\gamma > \gamma_1$. Consequently, for all $\gamma$ larger than $\gamma_0$ and $\gamma_1$, $\xi_i + \delta e \in X_i(z^{F(\gamma)}) \cap F(\gamma)$ for all $i \in I$.

Note that $(\bar{z}, \xi_i + \delta e, \bar{x}_i) \notin G_i$. Accordingly to Assumption C(4) there exists $\gamma_2 \in \Gamma$ such that for all $\gamma > \gamma_2$, $(z^{F(\gamma)}, \xi_i + \delta e, x_i^{F(\gamma)}) \notin G_i$. Consequently, for $\gamma$ large enough both $\xi_i + \delta e$ and $x_i^{F(\gamma)}$ belong to $X_i(z^{F(\gamma)}) \cap F(\gamma)$ and $\xi + \delta e \succ_{i,z^{F(\gamma)}}^{F(\gamma)} x_i^{F(\gamma)}$ for all $i \in I$. This contradicts our previous claim that for some $i_0$, there does not exist $\xi_{i_0}^F \in X_{i_0}(z^F) \cap F$ such that $\xi_{i_0} \succ_{i_0,z^F}^F x_{i_0}^F$.

Since preferences are also convex (Assumption C(3)) we deduce that they are locally non-satiated.

## Proof of Lemma 12

We want to prove that for all $(p, z, t) \in S^e \times Z^e \times_+$, if $(y_j) \in A^e(\omega + te, z)$ it follows that $p(\sum_{j\in J} y_j + \omega + te) > 0$. By Propositions 13 and 14, there exists a continuous, positive, linear functional $\pi$ which extends $p$ to the whole space $L$ and $\pi \in \cap_{j\in J}\hat{\mathcal{N}}_{Y_j(z)}(y_j)$. Hence, $(\pi, z, t) \in S \times Z \times_+$ and $(y_j) \in A(\omega + te, z)$. Since $\sum_{j\in J} y_j + \omega + te$ belongs to $L(e)_+$ we get by Assumption SA $0 < \pi(\sum_{j\in J} y_j + \omega + te) = p(\sum_{j\in J} y_j + \omega + te)$. Hence, the Lemma is proved.